On Self-normality and Abnormality in the Alternating Groups

Ben Brewster
Department of Mathematical Sciences
Binghamton University
Binghamton, NY 13902
USA
ben@math.binghamton.edu

Michael B. Ward
Department of Mathematics
Western Oregon University
Monmouth, OR 97361
USA
wardm@wou.edu

Qinhai H. Zhang *
Department of Mathematics
Shanxi Teachers University
Linfen, Shanxi 041004
Peoples Republic of China
zhangqh@dns.sxtu.edu.cn

April 17, 2003

Running title: Abnormal Subgroups

*Portions of this paper come from the third author's Ph.D. dissertation at Binghamton University. The third author is supported by the NSF of China and the NSF of Shanxi Province.

1 Introduction

A subgroup H of a group G is called an *abnormal subgroup* of G provided $g \in \langle H, H^g \rangle$ for all $g \in G$ and *self-normalizing* in G if the normalizer of H in G is equal to H itself. A subgroup H of a group G is called a *pronormal subgroup* of G provided H and H^g are conjugate in $\langle H, H^g \rangle$ for all $g \in G$. Consider the following results about the relationship between self-normality and abnormality. The first is a routine application of the Sylow theorems and it provides an ample supply of abnormal subgroups.

Theorem 1.1 *Let G be a finite group. Any subgroup of G containing the normalizer of any Sylow subgroup of G is abnormal and self-normalizing in G.*

Theorem 1.2 *[3, p. 247] Let H be a subgroup of a group G. Then H is abnormal in G if and only if H is both self-normalizing and pronormal in G.*

Theorem 1.3 *[3, p. 251] Let G be a finite group. Then a subgroup H of G is abnormal in G if and only if*
(i) every subgroup of G containing H is self-normalizing in G, and
(ii) H is not contained in two distinct conjugate subgroups of G.

We are interested in Theorem 1.3, which was proved by Philip Hall. Taunt [8, 9.2.11] proved that if G is a finite solvable group, then condition (i) alone is sufficient. Huppert [5, VI] and Doerk and Hawkes [3, p. 251] discuss whether (i) alone is sufficient for arbitrary finite groups. The answer is "no." A. Feldman [4] provided a counterexample in the finite simple group $U_3(3)$. Thus, it seems reasonable to ask whether there are certain situations in nonsolvable groups in which (i) guarantees abnormality. The answer is "yes."

A subgroup H of a group G is called *second maximal* in G provided H is a maximal subgroup of all maximal subgroups of G containing H. If G is a simple group, then note (i) holds for a second maximal subgroup H if and only if H is self-normalizing. In [11], the third author proved that if H is second maximal in the alternating group A_p where p is a prime, then H is abnormal in A_p if and only if H is self-normalizing. Therefore, (i) is sufficient to ensure abnormality for second maximal subgroups of A_p. In this paper, our goal is the following generalization.

Theorem *If H is a second maximal intransitive subgroup of the alternating group A_n where p is a prime, then H is abnormal in A_n if and only if H is self-normalizing.*

Note the added assumption of intransitivity.

Following some preliminaries in Section 2, we analyze in Section 3 a second maximal subgroup of A_n which is the intersection of an intransitive maximal subgroup with an imprimitive maximal subgroup. Section 4 contains our main result.

After completing this work, we learned of the unpublished thesis of A. Basile [1], which gives information about the second maximal subgroups of A_n. For portions of Section 3, we could cite Basile's work. Inasmuch as the thesis is not published and our proofs are not long, we opt instead to include self-contained proofs of the specific properties we need.

2 Preliminaries

Most of our permutation group notation and terminology is standard as, for example, in [2]. The following, however, may deserve clarification.

We use the term "imprimitive" to mean transitive, but not primitive. A "system of imprimitivity" is our name for what is also called a complete (nontrivial) block system of an imprimitive group.

Let $\Omega = \{1, 2, \ldots, n\}$. For any $G \leq S_n$ and $\Gamma \subseteq \Omega$, we define $G_\Gamma := \{\gamma \in G : a^\gamma \in \Gamma$ for every $a \in \Gamma\}$ (the *setwise stabilizer* of Γ) and $G_{(\Gamma)} := \{\gamma \in G : a^\gamma = a$ for every $a \in \Gamma\}$ (the *pointwise stabilizer* of Γ). For any subset $\Sigma := \{\Delta_1, \Delta_2, \ldots, \Delta_\ell\}$ of the power set of Ω, we define $G_\Sigma := \{\gamma \in G : \Delta^\gamma \in \Sigma$ for every $\Delta \in \Sigma\}$ and $G_{(\Sigma)} := \{\gamma \in G : \Delta^\gamma = \Delta$ for every $\Delta \in \Sigma\}$. Thus, $G_{(\Sigma)}$ fixes (setwise) the sets in Σ while G_Σ permutes the sets in Σ.

If $\Gamma = \{a\}$, we follow the usual convention and write G_a for G_Γ.

(When Σ is a partition of Ω, some authors write $G(\Delta_1, \Delta_2, \ldots, \Delta_\ell)$ for our $G_{(\Sigma)}$. If, in addition, all the sets in Σ have the same cardinality, then G_Σ is written by some as $G(\Delta_1 \wr \Delta_2 \wr \ldots \wr \Delta_\ell)$.)

The following lemma is well-known and easy to prove.

Lemma 2.1 *If G, Γ and Σ are as above, then for any $\alpha \in S_n$, $(G_\Gamma)^\alpha = G^\alpha_{\Gamma^\alpha}$, $\left(G_{(\Gamma)}\right)^\alpha = G^\alpha_{(\Gamma^\alpha)}$, $(G_\Sigma)^\alpha = G^\alpha_{\{\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_\ell^\alpha\}}$ and $\left(G_{(\Sigma)}\right)^\alpha = G^\alpha_{(\{\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_\ell^\alpha\})}$.*

If it happens that α normalizes G, then $(G_\Gamma)^\alpha = G_{\Gamma^\alpha}$, $\left(G_{(\Gamma)}\right)^\alpha = G_{(\Gamma^\alpha)}$, $(G_\Sigma)^\alpha = G_{\{\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_\ell^\alpha\}}$ and $\left(G_{(\Sigma)}\right)^\alpha = G_{(\{\Delta_1^\alpha, \Delta_2^\alpha, \ldots, \Delta_\ell^\alpha\})}$.

We sometimes use Lemma 2.1 without explicit reference. Other famous results on permutation groups are similarly slighted. We will also use the following classical results in our proof.

Proposition 2.2 *[2, Theorem 3.3A] Let H be a primitive subgroup of S_n. If H contains a 3-cycle, then $A_n \leq H$. If H contains a 2-cycle, then $H = S_n$.*

Proposition 2.3 *[10, Propostion 8.6] If H is a primitive subgroup of S_n and $a, b \in \{1, 2, \ldots, n\}$ with $a \neq b$, then either $H_a \neq H_b$ or H is regular of prime degree.*

Proposition 2.4 *[10, Propostion 17.5] Suppose H is a transitive subgroup of S_n, $a \in \{1, 2, \ldots, n\}$ and the lengths of the orbits of H_a are $1 = n_1 \leq n_2 \leq \cdots \leq n_k$. If there is an index $j > 1$ such that n_j and the maximal orbit length n_k are relatively prime, then H is imprimitive or H is regular of prime degree.*

Proposition 2.5 *Let H be an intransitive subgroup of A_n, $n \geq 4$ having exactly two orbits Δ_1 and Δ_2 with $|\Delta_1| \neq |\Delta_2|$. Then $(A_n)_{(\{\Delta_1, \Delta_2\})}$ is the unique intransitive maximal subgroup of A_n containing H.*

Proof: Clearly, $H \leq (A_n)_{(\{\Delta_1, \Delta_2\})}$, which is well-known to be a maximal subgroup of A_n (see, for example, [6] or [2, Exercises 5.2.8 and 5.2.9]).

Suppose $H \leq M$ where M is an intransitive maximal subgroup of A_n. Since orbits of H are contained in orbits of M, it is not hard to show that the orbits of M are Δ_1 and Δ_2. Therefore, $M \leq (A_n)_{(\{\Delta_1, \Delta_2\})}$. By maximality, $M = (A_n)_{(\{\Delta_1, \Delta_2\})}$, as was to be shown.

Our last preliminary result is not hard to prove, but it is perhaps not well-known.

Theorem 2.6 *[9, Theorem 8.19] Let H and K be normal subgroups of G such that $G = H \times K$ and let π and ρ be the corresponding projections of G onto H and K, respectively. Let $L \leq G$. Then $(H \cap L)$ is a normal subgroup of $L^\pi \leq H$, $(K \cap L)$ is a normal subgroup of $L^\rho \leq K$ and $L^\pi/(H \cap L) \cong L^\rho/(K \cap L)$.*

One final comment, all of our direct products are internal.

3 A Certain Second Maximal Subgroup of A_n

In this section, we give information about the structure and embedding of a second maximal subgroup of A_n which is the intersection of an intransitive maximal subgroup with an imprimitive maximal subgroup.

Lemma 3.1 *Assume $K \leq S_n$ contains an odd permutation. If $K \cap A_n$ is second maximal in A_n, then K is either maximal or second maximal in S_n.*

Proof: If K is neither maximal nor second maximal in S_n, then there exist subgroups M and L such that $K < M < L < S_n$. Since K contains an odd permutation, it follows that $|K \cap A_n| = |K|/2 < |M \cap A_n| = |M|/2 < |L \cap A_n| = |L|/2 < |A_n|$. Thus, $K \cap A_n < M \cap A_n < L \cap A_n < A_n$, contradicting the second maximality of $K \cap A_n$.

Theorem 3.2 *Let $\Omega = \{1, 2, \ldots, n\}$ and let $\Phi = \{\Delta_1, \Delta_2\}$ and $\Sigma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell\}$ be partitions of Ω with $|\Delta_1| < |\Delta_2|$, $\ell > 1$ and there is an integer $m > 1$ such that $|\Gamma_i| = m$ for all i. Furthermore, let $M_1 = (S_n)_{(\Phi)}$, $M_2 = (S_n)_\Sigma$, $K = M_1 \cap M_2$ and $H = M_1 \cap M_2 \cap A_n$.*

If H is second maximal in A_n, then

(i) K is second maximal in S_n;

(ii) $\ell \geq 3$ and, by reindexing Σ if necessary, $\Delta_1 = \Gamma_1$;

(iii) $K = (S_n)_{(\Delta_2)} \times \Big((S_n)_{(\Delta_1)}\Big)_{\{\Gamma_2,\Gamma_3,\ldots,\Gamma_\ell\}}$;

(iv) the orbits of H on Ω are Δ_1 and Δ_2; and

(v) if $H < L < A_n$ with $L \neq M_1 \cap A_n$ and $L \neq M_2 \cap A_n$, then L is primitive.

Proof: (i) We see $K < M_1$, for otherwise, $H = M_1 \cap A_n$ is a maximal subgroup of A_n (as in the proof of Proposition 2.5). Therefore, K is not a maximal subgroup of A_n.

Using Lemma 3.1, it suffices to show K contains a transposition. We claim there exist i, j with $|\Delta_i \cap \Gamma_j| \geq 2$. Suppose to the contrary $|\Delta_i \cap \Gamma_j| \leq 1$ for all i, j. For any j, we know $\Gamma_j = (\Delta_1 \cap \Gamma_j) \cup (\Delta_2 \cap \Gamma_j)$ and $|\Gamma_j| \geq 2$. Thus, $|\Delta_i \cap \Gamma_j| = 1$ for all i and j, but then $|\Delta_1| = \sum_{j=1}^{\ell} |\Delta_1 \cap \Gamma_j| = \sum_{j=1}^{\ell} |\Delta_2 \cap \Gamma_j| = |\Delta_2|$. That is contrary to our hypothesis. Therefore, $|\Delta_i \cap \Gamma_j| \geq 2$ for some i, j.

Let $a, b \in \Delta_i \cap \Gamma_j$ with $a \neq b$, then $(a\ b) \in M_1 \cap M_2 = K$ and the proof of (i) is complete.

(ii) First, we will show that for all j, either $\Gamma_j \cap \Delta_1$ or $\Gamma_j \cap \Delta_2$ is empty, in other words, either $\Gamma_j \subseteq \Delta_1$ or $\Gamma_j \subseteq \Delta_2$. Suppose $\Gamma_j \cap \Delta_1$ and $\Gamma_j \cap \Delta_2$ are both nonempty for some j. Let $t_i \in \Gamma_j \cap \Delta_i$ for $i = 1, 2$, then the transposition $\tau := (t_1\ t_2) \in M_2 \backslash M_1$. Thus, $K < \langle \tau, K \rangle \leq M_2$. By (i), $\langle \tau, K \rangle = M_2$. In particular, $\langle \tau, K \rangle$ is transitive on Σ. Since $\Gamma_i^\tau = \Gamma_i$ for each i, it follows that K is transitive on Σ.

Thus, for each i, there is $\alpha \in K$ such that $\Gamma_i^\alpha = \Gamma_1$ and so $(\Gamma_i \cap \Delta_1)^\alpha = \Gamma_i \cap \Delta_1$. That implies $|\Gamma_i \cap \Delta_1| = |\Gamma_1 \cap \Delta_1|$ for each i. Similarly, $|\Gamma_i \cap \Delta_2| = |\Gamma_1 \cap \Delta_2|$ for each i. Therefore, $|\Delta_k| = |\bigcup_{i=1}^{\ell} (\Gamma_i \cap \Delta_k)| = \sum_{i=1}^{\ell} |\Gamma_i \cap \Delta_k| = \ell |\Gamma_1 \cap \Delta_k|$ for $k = 1, 2$. Since $\Delta_1 < \Delta_2$, we cancel ℓ to obtain $1 \leq |\Gamma_i \cap \Delta_1| = |\Gamma_1 \cap \Delta_1| < |\Gamma_1 \cap \Delta_2| = |\Gamma_i \cap \Delta_2|$ for all i.

Let $s_i \in \Gamma_i \cap \Delta_1$ for each i and consider the permutation $\sigma = (s_1\ s_2\ \cdots\ s_\ell)$. Since $|\Gamma_i \cap \Delta_2| > 1$ for each i, we see $\sigma \in M_1 \backslash M_2$. Therefore, $K < \langle \sigma, K \rangle \leq M_1 < S_n$ and so $\langle \sigma, K \rangle = M_1$ by the second maximality of K. However, let u_{11} and u_{12} be distinct elements of $\Gamma_1 \cap \Delta_2$ and let $u_2 \in \Gamma_2 \cap \Delta_2$, then $\mu := (u_{11}\ u_2) \in M_1 = \langle \sigma, K \rangle$. Since σ fixes each $\Gamma_i \cap \Delta_2$ pointwise, it follows that $\{u_2, u_{12}\} = \{u_{11}, u_{12}\}^\mu = \{u_{11}, u_{12}\}^\beta \subseteq \Gamma^\beta$ for some $\beta \in K$, but $K \subseteq M_2$ so $\Gamma_1^\beta = \Gamma_m$ for some m. That is a contradiction because $u_2 \in \Gamma_2$ while $u_{12} \in \Gamma_1$.

Therefore, we have shown that for each j, either $\Gamma_j \subseteq \Delta_1$ or $\Gamma_j \subseteq \Delta_2$. By reindexing if necessary, we may suppose that $\Gamma_1, \ldots, \Gamma_t \subseteq \Delta_1$ and $\Gamma_{t+1}, \ldots, \Gamma_\ell \subseteq \Delta_2$. For the rest of the proof, it is notationally convenient to set $S := S_n$ and $A := A_n$.

By looking at the disjoint cycle decomposition of any element of K, we see $K = K_{(\Delta_2)} K_{(\Delta_1)} \leq S_{(\Delta_2)} K_{(\Delta_1)} \leq S_{(\Delta_2)} S_{(\Delta_1)} \leq M_1$. The first inclusion is proper if $t \neq 1$. The second inclusion is proper if $t \neq \ell - 1$. Since K is maximal in M_1 by (i), both inclusions cannot be proper. Moreover, $|\Delta_1| < |\Delta_2|$ implies $t \neq \ell - 1$ and also $\ell \neq 2$. Thus, $t = 1$ and $\ell \geq 3$.

(iii) Since we now have $\Gamma_1 = \Delta_1$, by looking again at the disjoint cycle decomposition of any element of K we can see $K \leq S_{(\Delta_2)} \times \left(S_{(\Delta_1)}\right)_{\{\Gamma_2, \Gamma_3, \ldots, \Gamma_\ell\}}$. Furthermore, $S_{(\Delta_2)}$ and $\left(S_{(\Delta_1)}\right)_{\{\Gamma_2, \Gamma_3, \ldots, \Gamma_\ell\}}$ are each clearly in $M_1 \cap M_2 = K$, so (iii) holds.

(iv) It suffices to show H is transitive on Δ_1 and Δ_2. Let $u, v \in \Delta_1$ with $u \neq v$. Take any $r, s \in \Gamma_2$ with $r \neq s$, then $(u\ v)(r\ s) \in H$ which takes u to v. Therefore, H is transitive on Δ_1.

Let $x, y \in \Delta_2$, then $x \in \Gamma_i$, $y \in \Gamma_j$ for some $i, j \geq 2$. Since $|\Gamma_i| = |\Gamma_j|$, it is easy to construct a permutation σ so that $x^\sigma = y$, $\Gamma_i^\sigma = \Gamma_j$, $\Gamma_j^\sigma = \Gamma_i$ and σ fixes the elements of Γ_k for each $k \neq i, j$. With $(u\ v)$ as above, either σ or $\sigma(u\ v)$ (depending on whether σ is even or odd) is an element of H taking x to y. Thus, H is transitive on Δ_2.

(v) Suppose $H < L < A_n$, L is maximal in A_n, $L \neq M_1 \cap A_n$ and $L \neq M_2 \cap A_n$. By (iv) and Proposition 2.5, $M_1 \cap A_n$ is the only intransitive subgroup of A_n containing H. Therefore, L is transitive.

Assume L is imprimitive. Let $T = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_s\}$ be a system of imprimitivity for L and let $M_3 = (S_n)_T$. Now $H \leq M_1 \cap L < L \leq M_3 \cap A_n < A_n$ with the second inclusion being proper because L is transitive while $M_1 \cap L$ is not. The second maximality of H implies $H = M_1 \cap L$ and $L = M_3 \cap A_n$. Thus, $H = M_1 \cap M_3 \cap A_n$.

Since T is a system of imprimitivity, there is an integer r such that $|\Lambda_i| = r \geq 2$ for each i and $s > 1$. The hypotheses of this theorem are therefore satisfied with Σ replaced by T and M_2 replaced by M_3. We apply (i)-(iii) and reindex T if necessary to conclude $\Gamma_1 = \Delta_1 = \Lambda_1$ and, consequently, $|\Omega| = \ell|\Gamma_1| = s|\Lambda_1|$ implies $\ell = s$.

Now suppose $\{\Gamma_2, \ldots, \Gamma_\ell\} \neq \{\Lambda_2, \ldots, \Lambda_\ell\}$. Without loss of generality, $\Gamma_2 \neq \Lambda_j$ for any j. Let $u \in \Gamma_2$. There exists i such that $u \in \Lambda_i$. Since $\Gamma_2 \neq \Lambda_i$ but $|\Gamma_2| = |\Lambda_i|$, there exists $v \in \Lambda_i \backslash \Gamma_2$. Take any $x, y \in \Gamma_1$ with $x \neq y$, then $(x\ y)(u\ v) \in M_1 \cap M_3 \cap A_n = H \leq M_2$. That is a contradiction because $(x\ y)(u\ v)$ moves exactly one element, namely u, of Γ_2 and, hence, does not take Γ_2 to Γ_k for any k. Therefore, $\{\Gamma_2, \ldots, \Gamma_\ell\} = \{\Lambda_2, \ldots, \Lambda_\ell\}$, $M_2 = M_3$ and $L = M_2 \cap A_n$, a contradiction. Thus, L is primitive as claimed.

Corollary 3.3 *Suppose H is a second maximal imprimitive subgroup of A_n having exactly two orbits Δ_1 and Δ_2 with $|\Delta_1| \neq |\Delta_2|$. Further suppose H is not contained in any proper primitive subgroup of A_n, then H is contained in at most two maximal subgroups of A_n and these two subgroups are not conjugate in A_n.*

If, in addition, H is self-normalizing, then H is abnormal in A_n.

Proof: Let $\Phi := \{\Delta_1, \Delta_2\}$, $S := S_n$ and $A = A_n$. Without loss of generality, assume $|\Delta_1| < |\Delta_2|$.

By Proposition 2.5, $A_{(\Phi)}$ is the unique intransitive maximal subgroup of A containing H. Assume there exists another maximal subgroup W of A containing H. By hypothesis, W is imprimitive. Let $\Sigma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell\}$ be a system of imprimitivity for W, then, by definition, for some $m > 1$, $|\Gamma_i| = m$ for all i, $\ell > 1$ and Σ is a partition of Ω.

Let $M_1 = S_{(\Phi)}$ and $M_2 = S_\Sigma$, then $W \leq M_2$ and $H \leq M_1 \cap M_2 \cap A < M_1 \cap A < A$. By second maximality, $H = M_1 \cap M_2 \cap A$. The hypotheses of Theorem 3.2 are therefore satisfied. By (v) and our hypotheses on H, H is contained in, at most, the two maximal subgroups $M_1 \cap A$ and $M_2 \cap A$ of A. Those two subgroups are not conjugate because the first is intransitive and the second is transitive.

If H is also self-normalizing, then H is abnormal in A by P. Hall's characterization of abnormal subgroups, Theorem 1.3.

4 The Main Theorem

Theorem 4.1 *Suppose H is a subgroup of A_n which is second maximal, self-normalizing and intransitive, then either H is abnormal in A_n or else H has exactly two orbits, which have cardinalities 2 and $n-2$.*

Proof: Let $\Phi := \{\Delta_1, \Delta_2, \ldots, \Delta_k\}$ be the set of orbits of H. By hypothesis, $k > 1$. There are the following cases to consider: (1) $k > 3$, (2) $k = 3$ and there are two orbits of equal cardinality greater than 1, (3) $k = 3$ and there are two orbits of cardinality equal to 1, (4) $k = 3$ and the orbits have distinct cardinalities, (5) $k = 2$ and both orbits have cardinality greater than 2, (6) $k = 2$ and there is an orbit of cardinality 1, (7) $k = 2$ and there is an orbit of cardinality 2. We will show each of cases (1)-(6) either do not occur or else implies H is abnormal in A_n. In a few cases, we mimic proofs from [7].

As usual, we let $S := S_n$ and $A := A_n$. We may assume $n \geq 5$ since A_2, A_3 and A_4 have no subgroups satisfying the hypotheses.

Case (1) H has more than three orbits.

By second maximality, $H \neq 1$. Without loss of generality, $|\Delta_1| > 1$. It is then routine to show $H \leq A_{(\Phi)} < A_{(\{\Delta_1 \cup \Delta_2, \ldots, \Delta_k\})} < A_{(\{\Delta_1 \cup \Delta_2 \cup \Delta_3, \ldots, \Delta_k\})} < A$, contradicting the second maximality of H.

Case (2) H has three orbits and, without loss of generality, $|\Delta_1| = |\Delta_2| > 1$.

In this case, $H \leq A_{(\Phi)} < \left(A_{(\Phi)}\right)_{\{\Delta_1,\Delta_2\}} < A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})} < A$, contradicting second maximality.

Case (3) H has three orbits and, without loss of generality, $|\Delta_1| = |\Delta_2| = 1$.

Let $\Delta_1 = \{1\}$ and $\Delta_2 = \{2\}$. We have $H \leq A_{(\Phi)} < A_\Phi < A$, so $H = A_{(\Phi)}$ by second maximality. (In other words, H is a 2-point stabilizer in A, which is isomorphic to A_{n-2}.) The permutation $(1\ 2)(3\ 4) \in A$ therefore normalizes H (by Lemma 2.1) but is not in H, contradicting the self-normality hypothesis.

Case (4) H has three orbits and, without loss of generality, $|\Delta_1| < |\Delta_2| < |\Delta_3|$.

In this case, it is not hard to show $H \leq A_{(\Phi)} < A_{(\{\Delta_1,\Delta_2\cup\Delta_3\})} < A$. Thus, $H = A_{(\Phi)}$. Furthermore, if $|\Delta_1 \cup \Delta_2| = |\Delta_3|$, then $H = A_{(\Phi)} < A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})} < A_{\{\Delta_1\cup\Delta_2,\Delta_3\}} < A$, a contradiction. Therefore, we may assume $|\Delta_1 \cup \Delta_2| \neq |\Delta_3|$.

Since $|\Delta_3| \geq 3$, H contains a 3-cycle. By Proposition 2.2, H is not contained in a maximal primitive subgroup of A. Assume H is contained in a maximal imprimitive subgroup M of A. Thus, $H \leq A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})} \cap M < M < A$, the second inclusion being proper since M is transitive but the intersection is not. Therefore, $H = A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})}$. However, we can show the hypotheses of Theorem 3.2 are satisfied with $M_1 = A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})}$ and $M_2 = M$. Therefore, by Theorem 3.2(iv), H has only two orbits, a contradiction. Hence, H is not contained in a maximal imprimitive subgroup of A.

Thus, the only proper subgroups of A which could properly contain H are the intransitive subgroups $A_{(\{\Delta_1\cup\Delta_2,\Delta_3\})}$, $A_{(\{\Delta_1,\Delta_2\cup\Delta_3\})}$ and $A_{(\{\Delta_1\cup\Delta_3,\Delta_3\})}$. No two of those are conjugate because our hypotheses on orbit cardinalities and our reduction to the case $|\Delta_1 \cup \Delta_2| \neq |\Delta_3|$ show that no two pairs of sets in any of those three stabilizers have matching cardinalities. Therefore, H is abnormal in A by Philip Hall's characterization, Theorem 1.3.

Case (5) H has two orbits Δ_1 and Δ_2 each having cardinality greater than 2.

First assume $|\Delta_1| = |\Delta_2|$. Here we have $H \leq A_{(\Phi)} < A_\Phi < A$. Therefore, $H = A_{(\Phi)}$. However, $A_{(\Phi)}$ is a normal subgroup of A_Φ as is easily shown by direct calculation (or the reader who notes the wreath action of A_Φ on Ω will see $[A_\Phi : A_{(\Phi)}] = 2$). Thus, H is not self-normalizing, contrary to hypothesis. Therefore, we may assume $|\Delta_1| \neq |\Delta_2|$.

For any $\sigma \in H$, from the disjoint cycle decomposition of σ, we see there exist unique $\sigma_1 \in S_{(\Delta_2)}$, $\sigma_2 \in S_{(\Delta_1)}$ with $\sigma = \sigma_1\sigma_2$. Thus, $H \leq S_{(\Delta_2)} \times S_{(\Delta_1)}$ and we define $H_i := \{\sigma_i : \sigma \in H\}$ for $i = 1, 2$. In other words, H_1 is the projection of H onto $S_{(\Delta_2)}$ and H_2 is the projection of H onto $S_{(\Delta_1)}$. Therefore, $H_1 \leq S_{(\Delta_2)}$ and $H_2 \leq S_{(\Delta_1)}$.

If $H_1 < S_{(\Delta_2)}$, then $H \leq (H_1 \times S_{(\Delta_1)}) \cap A < A_{(\Phi)}$ with the last inclusion being proper because we can take $\alpha \in S_{(\Delta_2)} \backslash H_1$ and find $\beta \in S_{(\Delta_1)}$ such that $\alpha\beta \in A$. Therefore, $\alpha\beta \in A_{(\Phi)}$ but $\alpha\beta \notin (H_1 \times S_{(\Delta_1)}) \cap A$. By the second maximality of H, $H = (H_1 \times S_{(\Delta_1)}) \cap A$. Since $|\Delta_2| > 2$, it follows that H contains a 3-cycle from $S_{(\Delta_1)}$. Therefore, by Proposition 2.2, H is not contained in any proper primitive subgroup of A. By Corollary 3.3, H is abnormal.

Hence, we may assume $H_1 = S_{(\Delta_2)}$ and, by symmetry, $H_2 = S_{(\Delta_1)}$. Let $N_1 = H_{(\Delta_2)} \leq A_{(\Delta_2)}$ and $N_2 = H_{(\Delta_1)} \leq A_{(\Delta_1)}$. It is routine to show N_1 is a normal subgroup of H_1 and N_2 is a normal subgroup of H_2.

If $N_1 = A_{(\Delta_2)}$ or $N_2 = A_{(\Delta_1)}$, then H contains a 3-cycle and we are finished as above. Thus, N_1 is a normal subgroup of $S_{(\Delta_2)} \cong S_{|\Delta_1|}$ properly contained in $A_{(\Delta_2)} \cong A_{|\Delta_1|}$. Therefore, either $N_1 = 1$ or $|\Delta_1| \in \{3, 4\}$. Similarly, either $N_2 = 1$ or $|\Delta_2| \in \{3, 4\}$.

Recalling that $H \leq S_{(\Delta_2)} \times S_{(\Delta_1)} = H_1 \times H_2$ and noting $N_1 = H \cap S_{(\Delta_2)}$ and $N_2 = H \cap S_{(\Delta_1)}$, a direct application of Theorem 2.6 gives $H_1/N_1 \cong H_2/N_2$.

If $N_1 = 1 = N_2$, then from the isomorphism we obtain $|\Delta_1|! = |H_1| = |H_2| = |\Delta_2|!$, contradicting our assumption $|\Delta_1| \neq |\Delta_2|$. If $N_1 = 1 \neq N_2$, then $|\Delta_1|! = |H_1| = |H_2/N_2| \leq 4!$. So $|\Delta_1| \in \{3, 4\}$. Likewise, if $N_2 = 1 \neq N_1$, then $|\Delta_2| \in \{3, 4\}$. Thus, in all cases, $|\Delta_1|, |\Delta_2| \in \{3, 4\}$. Therefore, $n = |\Delta_1| + |\Delta_2| = 7$. We can assume $|\Delta_1| = 3$ and $|\Delta_2| = 4$. It follows that $N_1 = 1$ and $|N_2| = 4$.

We claim $|H| = 24$. Once we establish the claim, we are finished because then H contains a Sylow 2-subgroup of A_7, which is self-normalizing in A_7. Therefore, H is abnormal by Theorem 1.1.

Let $\sigma_2 \in H_2$, then there exists $\sigma_1 \in H_1$ with $\sigma_1\sigma_2 \in H$. Suppose $\sigma_1' \in H_1$ with $\sigma_1'\sigma_2 \in H$, then $(\sigma_1'\sigma_2)(\sigma_1\sigma_2)^{-1} = \sigma_1'\sigma_1^{-1} \in H \cap S_{(\Delta_2)} = N_1 = 1$. Therefore, σ_1 is unique and so the relation taking σ_2 to $\sigma_1\sigma_2$ is a function from H_2 to H. That function is injective by the uniqueness we noted when defining H_1 and H_2. Therefore, $24 = |H_2| \leq |H|$. However, we also have H is a maximal subgroup of $A_{(\Phi)}$, which has order 72 in this situation. Thus, $|H| = 24$ as claimed.

Case (6) H has exactly two orbits and, without loss of generality, $|\Delta_1| \leq 2$.

As usual, $A_{(\Phi)}$ is the unique intransitive subgroup of A containing H. In this case, $A_{(\Phi)}$ is simply A_{Δ_1}.

We will show H is abnormal in A directly. Let $g \in A$ and let $M := \langle H, H^g \rangle$. We need to show $g \in M$. Either $M = A$, $M = H$ or M is a maximal subgroup of A. The first situation is trivial. In the second case, $H = H^g$, so $g \in H$ since H is self-normalizing. Therefore, we may assume M is a maximal subgroup of A.

First we will handle the cases where M is intransitive and where $|\Delta_1| = 1$. To finish, we will look at M being imprimitive and then primitive.

If M is intransitive, then $M = A_{\Delta_1}$, by the uniqueness noted above. Thus, $H = (H^g)^{g^{-1}} \leq A_{\Delta_1^{g^{-1}}}$ which implies $A_{\Delta_1} = A_{\Delta_1^{g^{-1}}}$, again by uniqueness. If follows that $\Delta_1^{g^{-1}} = \Delta_1$ and $g \in A_{\Delta_1} = M$, as was to be shown.

Assume M is transitive, then $H \leq M \cap A_{\Delta_1} = M_{\Delta_1} < M < A$. Therefore, $H = M_{\Delta_1}$. Similarly, $H^g = M_{\Delta_1^g}$ since H^g is also second maximal in A.

In all the remaining cases, our stategy is to show $\Delta_1^g = \Delta_1^x$ for some $x \in M$. For if that is the case, then $H^g = M_{\Delta_1^g} = M_{\Delta_1^x} = (M_{\Delta_1})^x = H^x$. Because of the self-normality, $gx^{-1} \in H \leq M$ and so $g \in M$, as was to be shown..

If $|\Delta_1| = 1$, then by transitivity, $\Delta_1^g = \Delta_1^x$ for some $x \in M$. We may henceforth assume $|\Delta_1| = 2$.

Next consider the case where M is imprimitive. Let Σ be a system of imprimitivity for M, then $M = A_\Sigma$, by maximality. Thus, $H = M_{\Delta_1} = M \cap A_{\Delta_1} = A_\Sigma \cap A_{\Delta_1} = S_\Sigma \cap S_{\Delta_1} \cap A$, which means the hypotheses of Theorem 3.2 are satisfied. By Theorem 3.2, $\Delta_1 \in \Sigma$. We showed earlier that $H^g = M_{\Delta_1^g}$ and so we have $\Delta_1^g \in \Sigma$ by applying Theorem 3.2 to H^g. Therefore, there exists $x \in M$ such that $\Delta_1^g = \Delta_1^x$.

Finally, assume M is primitive. Let $a \in \Delta_1$ and $c \in \Delta_2$. Since $|\Delta_1| = 2$, we have $H_a = (M_{\Delta_1})_a = M_{(\Delta_1)}$ and $[H : H_a] = 2$. Thus, either $H_aH_c = H$, in which case H_a is transitive on Δ_2, or else $H_aH_c = H_a$, in which case H_a has two orbits of lengths $(n-2)/2$ on Δ_2.

Consider M_a. If M_a fixes some point other than a, then M is regular of prime degree by Proposition 2.3, a contradiction. Combining that with the fact that M_a orbits on $\{1, 2, \ldots n\}$ are unions of H_a orbits, we see that the M_a orbits have lengths 1 and $n-1$ or else 1, $(n-2)/2$ and $n/2$. In the latter case, M would again by regular of prime degree by Proposition 2.4.

Therefore, the first case holds and M is 2-transitive. Once again, there exists $x \in M$ such that $\Delta_1^g = \Delta_1^x$ and the proof is complete.

References

[1] A. Basile. Second Maximal Subgroups of the Finite Alternating and Symmetric Groups. Ph.D. thesis, The Australian National University (2001).

[2] J. D. Dixon and B. Mortimer. *Permutation Groups* (Springer-Verlag, 1996).

[3] A. Doerk and T. Hawkes. *Finite Soluble Groups* (Walter de Gruyter, 1992).

[4] A. Feldman, A nonabnormal subgroup contained only in self-normalizing subgroups in a finite group, *Arch. Math.* **70** (1998), 9-10.

[5] B. Huppert. *Endliche Gruppen I* (Springer-Verlag, 1967).

[6] M. W. Liebeck, C. E. Praeger, J. Saxl. A classification of the maximal subgroups of the finite alternating and symmetric groups. *J. Algebra* **111** (1987), 365-383.

[7] P. P. Pálfy. On Feit's example of intervals in subgroup lattices. *J. Algebra* **116** (1988), 471-479.

[8] D. J. S. Robinson. *A Course in the Theory of Groups* (Springer-Verlag, 1982).

[9] J. S. Rose. *A Course on Group Theory* (Cambridge University Press, 1978).

[10] H. Wielandt. *Finite Permutation Groups* (Academic Press, 1964).

[11] Q. H. Zhang. On self-normality and abnormality in the alternating group A_p. *Arch. Math.* **72** (1999), 1-4.