Cayley-Sudoku tables: An undergraduate research project *Preliminary Report* (Slides for a talk given at the Joint Mathematics Meetings 2007, New Orleans)

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Definition 1 *A* **Sudoku table** *is an $n \times n$ array partitioned into rectangular blocks of some fixed size such that each of n symbols appear exactly once in each row, each column, and each block.*

In the standard Sudoku puzzle, the array is 9×9, the blocks are 3×3 and the symbols are the numbers 1 through 9.

Standard Example:

1	4	7	2	5	8	3	6	9
2	5	8	3	6	9	4	7	1
3	6	9	4	7	1	5	8	2
4	7	1	5	8	2	6	9	3
5	8	2	6	9	3	7	1	4
6	9	3	7	1	4	8	2	5
7	1	4	8	2	5	9	3	6
8	2	5	9	3	6	1	4	7
9	3	6	1	4	7	2	5	8

Definition 2 *A* **Cayley table** *is an operation table for a finite group.*

Definition 3 *A* **Cayley-Sudoku table** *is a Cayley table which is also a Sudoku table (excluding the group elements that label rows and columns in the Cayley table).*

Example 1 *A Cayley-Sudoku Table for $\mathbb{Z}_9$, with row and column labels shown for clarity.*

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

Example 2 *A Cayley-Sudoku table for* A_4

To save space, rename elements of A_4:

1:= 1	(123):= 5	(132):= 9
(12)(34):= 2	(243):= 6	(143):= 10
(13)(24):= 3	(142):= 7	(234):= 11
(14)(23):= 4	(134):= 8	(124):= 12

A_4	1	5	9	2	6	10	3	7	11	4	8	12
1	1	5	9	2	6	10	3	7	11	4	8	12
2	2	8	11	1	7	12	4	6	9	3	5	10
3	3	6	12	4	5	11	1	8	10	2	7	9
4	4	7	10	3	8	9	2	5	12	1	6	11
5	5	9	1	6	10	2	7	11	3	8	12	4
8	8	11	2	7	12	1	6	9	4	5	10	3
6	6	12	3	5	11	4	8	10	1	7	9	2
7	7	10	4	8	9	3	5	12	2	6	11	1
9	9	1	5	10	2	6	11	3	7	12	4	8
11	11	2	8	12	1	7	9	4	6	10	3	5
12	12	3	6	11	4	5	10	1	8	9	2	7
10	10	4	7	9	3	8	12	2	5	11	1	6

How are those examples constructed?

Study the layout of the Cayley tables.

Example 1 Revisited

Consider the subgroup $H := \langle 3 \rangle = \{0, 3, 6\}$. Its cosets are
$H + 0 = \{0, 3, 6\} = 0 + H = H$,
$H + 1 = \{1, 4, 7\} = 1 + H$, and
$H + 2 = \{2, 5, 8\} = 2 + H$.

The sets
$T_1 := \{0, 1, 2\}$, $T_2 := \{3, 4, 5\}$, $T_3 := \{6, 7, 8\}$ partition $\mathbb{Z}_9$ into complete sets of left coset representatives (AKA left transversals).

		H			$H+1$			$H+2$		
	$\mathbb{Z}_9$	0	3	6	1	4	7	2	5	8
	0	0	3	6	1	4	7	2	5	8
T_1	1	1	4	7	2	5	8	3	6	0
	2	2	5	8	3	6	0	4	7	1
	3	3	6	0	4	7	1	5	8	2
T_2	4	4	7	1	5	8	2	6	0	3
	5	5	8	2	6	0	3	7	1	4
	6	6	0	3	7	1	4	8	2	5
T_3	7	7	1	4	8	2	5	0	3	6
	8	8	2	5	0	3	6	1	4	7

Example 2 Revisited

Consider the subgroup $H :=< (123) >= \{1, (123), (132)\} = \{1, 5, 9\}$. Its cosets are

$$
\begin{array}{rr}
H1 = H = \{1,5,9\} & 1H = H = \{1,5,9\} \\
H(12)(34) = \{2,6,10\} & (12)(34)H = \{2,8,11\} \\
H(13)(24) = \{3,7,11\} & (13)(24)H = \{3,6,12\} \\
H(14)(23) = \{4,8,12\} & (14)(23)H = \{4,7,10\}
\end{array}
$$

The sets
$T_1 := \{1, 2, 3, 4\}$, $T_2 := \{5, 8, 6, 7\}$,
$T_3 := \{9, 11, 12, 10\}$ partition A_4 into complete sets of left coset representatives.

		H			$H(12)(34)$			$H(13)(24)$			$H(14)(23)$		
	A_4	1	5	9	2	6	10	3	7	11	4	8	12
T_1	1	1	5	9	2	6	10	3	7	11	4	8	12
	2	2	8	11	1	7	12	4	6	9	3	5	10
	3	3	6	12	4	5	11	1	8	10	2	7	9
	4	4	7	10	3	8	9	2	5	12	1	6	11
T_2	5	5	9	1	6	10	2	7	11	3	8	12	4
	8	8	11	2	7	12	1	6	9	4	5	10	3
	6	6	12	3	5	11	4	8	10	1	7	9	2
	7	7	10	4	8	9	3	5	12	2	6	11	1
T_3	9	9	1	5	10	2	6	11	3	7	12	4	8
	11	11	2	8	12	1	7	9	4	6	10	3	5
	12	12	3	6	11	4	5	10	1	8	9	2	7
	10	10	4	7	9	3	8	12	2	5	11	1	6

Theorem 1 (K. Schloeman) *For any finite group G and any subgroup H of G, let the distinct* <u>*right*</u> *cosets of H in G be $H = Hg_1$, Hg_2, . . . , Hg_m and let T_1, T_2, . . . , T_n be any partition of G into complete sets of* <u>*left*</u> *coset representatives of H in G. Labeling the rows and columns of a Cayley table for G as follows will give a Cayley-Sudoku table for G.*

	H	Hg_2	$\cdots$	Hg_m
T_1				
T_2				
$\vdots$				
T_n				

Theorem 1 (continued) *Also, if the distinct* *<u>left</u>* *cosets of* H *in* G *are* $H = g_1'H$, $g_2'H$, $\ldots$, $g_m'H$ *and* R_1, R_2, $\ldots$, R_n *is any partition of* G *into complete sets of* *<u>right</u>* *coset representatives of* H *in* G*, then the following is also a Cayley-Sudoku table for* G.

	R_1	R_2	$\cdots$	R_n
H				
$g_2'H$				
$\vdots$				
$g_m'H$				

Example 2 Re-revisited

Reminder	1:= 1	(123):= 5	(132):= 9
	(12)(34):= 2	(243):= 6	(143):= 10
	(13)(24):= 3	(142):= 7	(234):= 11
	(14)(23):= 4	(134):= 8	(124):= 12

Let $H := \{1, (12)(34), (13)(24), (14)(23)\} = \{1, 2, 3, 4\}$, the Klein 4-group, normal in A_4.

The other cosets are

$(123)H = 5H = \{5, 8, 6, 7\} = H5$ and

$(132)H = 9H = \{9, 11, 12, 10\} = H9$.

The sets R_1, R_2, R_3, R_4 are easily seen to be complete sets of right coset representatives.

	R_1			R_2			R_3			R_4		
A_4	1	5	9	2	6	10	3	7	11	4	8	12
H 1	1	5	9	2	6	10	3	7	11	4	8	12
2	2	8	11	1	7	12	4	6	9	3	5	10
3	3	6	12	4	5	11	1	8	10	2	7	9
4	4	7	10	3	8	9	2	5	12	1	6	11
$5H$ 5	5	9	1	6	10	2	7	11	3	8	12	4
8	8	11	2	7	12	1	6	9	4	5	10	3
6	6	12	3	5	11	4	8	10	1	7	9	2
7	7	10	4	8	9	3	5	12	2	6	11	1
$9H$ 9	9	1	5	10	2	6	11	3	7	12	4	8
11	11	2	8	12	1	7	9	4	6	10	3	5
12	12	3	6	11	4	5	10	1	8	9	2	7
10	10	4	7	9	3	8	12	2	5	11	1	6

Theorem 2 (J. Carmichael & M. Ward) *For groups of order 9, Keith's Theorem 1 produces all possible Cayley-Sudoku tables.*

Theorem 3 (Christmas Eve) *Suppose* $H = \{1 = h_1, h_2, \ldots, h_n\}$ *and* $K = \{1 = k_1, k_2, \ldots, k_m\}$ *are subgroups of a finite group* G *such that* $G = KH := \{k_i h_j : k_i \in K$ *and* $h_j \in H\}$ *and* $K \cap H = 1$. *Labeling the rows and columns of a Cayley table for* G *as follows will give a Cayley-Sudoku table for* G.

	H	k_2H	$\cdots$	k_mH
K				
Kh_2				
$\vdots$				
Kh_n				

Example 3 *A Cayley-Sudoku table for* S_4 *not of the type produced by Theorem 1*

Extend the numbering introduced earlier for A_4 to a numbering of elements of S_4 by defining $(12)\ell := \ell + 12$ for permutation number ℓ in A_4.

Let $H :=< (123) >= \{1, 5, 9\}$, a Sylow 3-subgroup of S_4, and
$K :=< (13), (1234) >= \{1, 2, 3, 4, 17, 18, 19, 20\}$, a Sylow 2-subgroup of S_4.

It is easy to check that $S_4 = KH$ and $K \cap H = 1$. Therefore, by Theorem. 3, the following is a Cayley-Sudoku table of S_4. By inspection, we see it is not of the type produced by Theorem. 1.

	S_4	H			$2H$			$3H$			$4H$		
		1	5	9	2	8	11	3	6	12	4	7	10
K	1	1	5	9	2	8	11	3	6	12	4	7	10
	2	2	8	11	1	5	9	4	7	10	3	6	12
	3	3	6	12	4	7	10	1	5	9	2	8	11
	4	4	7	10	3	6	12	2	8	11	1	5	9
	17	17	21	13	18	24	15	19	22	16	20	23	14
	18	18	24	15	17	21	13	20	23	14	19	22	16
	19	19	22	16	20	23	14	17	21	13	18	24	15
	20	20	23	14	19	22	16	18	24	15	17	21	13
$K5$	5	5	9	1	6	12	3	7	10	4	8	11	2
	8	8	11	2	7	10	4	6	12	3	5	9	1
	6	6	12	3	5	9	1	8	11	2	7	10	4
	7	7	10	4	8	11	2	5	9	1	6	12	3
	21	21	13	17	22	16	19	23	14	20	24	15	18
	24	24	15	18	23	14	20	22	16	19	21	13	17
	22	22	16	19	21	13	17	24	15	18	23	14	20
	23	23	14	20	24	15	18	21	13	17	22	16	19
$K9$	9	9	1	5	10	4	7	11	2	8	12	3	6
	11	11	2	8	12	3	6	9	1	5	10	4	7
	12	12	3	6	11	2	8	10	4	7	9	1	5
	10	10	4	7	9	1	5	12	3	6	11	2	8
	13	13	17	21	14	20	23	15	18	24	16	19	22
	15	15	18	24	16	19	22	13	17	21	14	20	23
	16	16	19	22	15	18	24	14	20	23	13	17	21
	14	14	20	23	13	17	21	16	19	22	15	18	24

Open Questions

- Are there ways to build Cayley-Sudoku tables without using cosets/coset representatives?

- If a column [row] of blocks is labeled by elements of a subgroup, must the other columns [rows] of blocks be labeled by cosets of that subgroup?

- If $A = \{1 = a_1, a_2, \ldots, a_n\}$ and $B = \{1 = b_1, b_2, \ldots, b_m\}$ are subsets of a finite group G such that the following table contains each element of G exactly once, must either A or B be a subgroup of G?

	1	b_2	$\cdots$	b_m
1				
a_2				
$\vdots$				
a_n				

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Keith Schloeman has now answered all the open questions on the previous slide. The answers are, respectively, "yes", "no" and "no".