Orthogonal & Magic Cayley-Sudoku Tables

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Speaker's Qualifications

Cayley-Sudoku Tables–Expert

Latin Squares, Orthogonality, Magic Squares–Dilettante

Cayley-Sudoku Tables

Definition

*A **Sudoku table** is an $n \times n$ array partitioned into rectangular blocks of some fixed size such that each of n symbols appear exactly once in each row, each column, and each block.*

(In the ubiquitous Sudoku puzzle, the array is 9×9; the blocks are 3×3; and the symbols are the numbers 1 through 9.)

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Any Cayley table is a (bordered) Latin square, so it is $2/3$ sudoku.

Definition

*A **Cayley-Sudoku table** [C-S Table] is a Cayley table which is also a (bordered) Sudoku table.*

Example: $\mathbb{Z}_9$

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

A Cayley-Sudoku Table for $\mathbb{Z}_9 := \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under addition mod 9

Example: D_4

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
()	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
(1,2)(3,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(2,4)	(1,3)(2,4)	(1,2,3,4)	()	(1,4,3,2)
(1,2,3,4)	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,3)	(1,2)(3,4)	(2,4)	(1,4)(2,3)
(2,4)	(2,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)	()
(1,3)(2,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,2,3,4)	(1,2)(3,4)	(2,4)	(1,4)(2,3)	(1,3)
(1,4)(2,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,3)	()	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)
(1,4,3,2)	(1,4,3,2)	()	(1,2,3,4)	(1,3)(2,4)	(2,4)	(1,4)(2,3)	(1,3)	(1,2)(3,4)
(1,3)	(1,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,2,3,4)	()	(1,4,3,2)	(1,3)(2,4)

A Cayley-Sudoku Table for D_4, dihedral group of order 8

How the $\mathbb{Z}_9$ Cayley-Sudoku table was made: Column Labels

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

Consider the subgroup $\langle 3 \rangle = \{0, 3, 6\}$. The column labels of each block $[0,3,6] = [0,3,6] + 0$, $[1,4,7] = [0,3,6] + 1$ and $[2,5,8] = [0,3,6] + 2$ are right cosets of the subgroup.

Row Labels

	0	3	6	1	4	7	2	5	8
0	0	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	0
2	2	5	8	3	6	0	4	7	1
3	3	6	0	4	7	1	5	8	2
4	4	7	1	5	8	2	6	0	3
5	5	8	2	6	0	3	7	1	4
6	6	0	3	7	1	4	8	2	5
7	7	1	4	8	2	5	0	3	6
8	8	2	5	0	3	6	1	4	7

The row labels of each block are complete sets of left coset representatives, i.e. one element from each left coset.

$0 + [0, 3, 6] = [0, 3, 6]$

$1 + [0, 3, 6] = [1, 4, 7]$

$2 + [0, 3, 6] = [2, 5, 8]$

Generalization

Theorem (J. Dénes 1967; J. Carmichael, K. Schloeman, M. Ward 2010)
Let G be a finite group. Assume H is a subgroup of G. Let $Hg_1, Hg_2, \ldots, Hg_n$ be the distinct right cosets of H in G and let $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives [CSLCR] of H in G. The following layout gives a Cayley-Sudoku table.

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

We call this a Cayley-Sudoku table based on H.

Cayley-Sudoku tables exist for every group based on any subgroup.

- In this talk, order matters. We regard cosets and CSLCR as (ordered) lists, denoted [ ].

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- Our Cayley-Sudoku tables are made this way.

	$[Hg_1]$	$[Hg_2]$	$\dots$	$[Hg_n]$
$[T_1]$				
$[T_2]$				
$\vdots$				
$[T_k]$				

Inspiration

In constructing mutually orthogonal sets of sudoku and magic sudoku tables, Pedersen & Vis (*College Math. J.* 2009) and Lorch (*Amer. Math. Monthly* 2012) re-rediscovered this construction (for finite fields, heavily disguised).

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- What about orthogonal Cayley-Sudoku tables?

Inspiration

In constructing mutually orthogonal sets of sudoku and magic sudoku tables, Pedersen & Vis (*College Math. J.* 2009) and Lorch (*Amer. Math. Monthly* 2012) re-rediscovered this construction (for finite fields, heavily disguised).

- What about orthogonal Cayley-Sudoku tables?
- In what sense might a Cayley-Sudoku table be magic?

Orthogonal Mates & An Old Question

Definition

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*Two Latin squares are **orthogonal**, or are **orthogonal mates**, provided each ordered pair of symbols occurs exactly once when the squares are superimposed.*

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Old Question

The Cayley (i.e. operation) table of any finite group is a (bordered) Latin square.

Which Cayley tables have orthogonal mates?

Example

These Cayley tables of $\mathbb{Z}_3 := \{0, 1, 2\}$ *under addition modulo 3 are orthogonal mates.*

	0	*1*	*2*
0	*0*	*1*	*2*
1	*1*	*2*	*0*
2	*2*	*0*	*1*

	0	*1*	*2*
0	*0*	*1*	*2*
2	*2*	*0*	*1*
1	*1*	*2*	*0*

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	0	*1*	*2*
0	*0*	*1*	*2*
2	*2*	*0*	*1*
1	*1*	*2*	*0*

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

00	11	22
12	20	01
21	02	10

Non-example

These Cayley tables of $\mathbb{Z}_3 := \{0,1,2\}$ *are not orthogonal mates.*

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

	0	2	1
0	0	2	1
2	2	1	0
1	1	0	2

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0	0	1	2
1	1	2	0
2	2	0	1

	0	2	1
0	0	2	1
2	2	1	0
1	1	0	2

0	1	2
1	2	0
2	0	1

0	2	1
2	1	0
1	0	2

00	12	21
12	21	00
21	00	12

Answers to the Old Question

Definition

A ***complete mapping*** *of a group* G *is a bijection* $\theta : G \to G$ *for which the mapping* $\eta : G \to G$ *where* $\eta(x) = x\theta(x)$ *is also a bijection.*

Answers to the Old Question

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Theorem

The Cayley table of a finite group G has an orthogonal mate iff G admits a complete mapping (i.e. G is admissible).

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Definition

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Theorem

The Cayley table of a finite group G has an orthogonal mate iff G admits a complete mapping (i.e. G is admissible).

Hall-Paige Conjecture (Theorem circa 2009)

A finite group G has a complete map iff G has trivial or non-cyclic Sylow 2-subgroups.

Orthogonal Cayley-Sudoku Tables?

Given a Cayley-Sudoku table based on a subgroup H is there an orthogonal mate that is also a Cayley-Sudoku tables based on H?

Example

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
()	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
(1,2)(3,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(2,4)	(1,3)(2,4)	(1,2,3,4)	()	(1,4,3,2)
(1,2,3,4)	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,3)	(1,2)(3,4)	(2,4)	(1,4)(2,3)
(2,4)	(2,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)	()
(1,3)(2,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,2,3,4)	(1,2)(3,4)	(2,4)	(1,4)(2,3)	(1,3)
(1,4)(2,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,3)	()	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)
(1,4,3,2)	(1,4,3,2)	()	(1,2,3,4)	(1,3)(2,4)	(2,4)	(1,4)(2,3)	(1,3)	(1,2)(3,4)
(1,3)	(1,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,2,3,4)	()	(1,4,3,2)	(1,3)(2,4)

	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
()	()	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	(1,3)	(1,2)(3,4)	(2,4)
(2,4)	(2,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)	()
(1,4,3,2)	(1,4,3,2)	()	(1,2,3,4)	(1,3)(2,4)	(2,4)	(1,4)(2,3)	(1,3)	(1,2)(3,4)
(1,2)(3,4)	(1,2)(3,4)	(1,3)	(1,4)(2,3)	(2,4)	(1,3)(2,4)	(1,2,3,4)	()	(1,4,3,2)
(1,4)(2,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,3)	()	(1,4,3,2)	(1,3)(2,4)	(1,2,3,4)
(1,2,3,4)	(1,2,3,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,3)	(1,2)(3,4)	(2,4)	(1,4)(2,3)
(1,3)	(1,3)	(1,4)(2,3)	(2,4)	(1,2)(3,4)	(1,2,3,4)	()	(1,4,3,2)	(1,3)(2,4)
(1,3)(2,4)	(1,3)(2,4)	(1,4,3,2)	()	(1,2,3,4)	(1,2)(3,4)	(2,4)	(1,4)(2,3)	(1,3)

Orthogonal Cayley-Sudoku Tables for D_4 based on $\{(), (1,2,3,4), (1,3)(2,4), (1,4,3,2)\}$.

“θ^{-1} Construction”

Observation (MW 2013)

Let G be a finite group. Assume H is a subgroup of G. Let $Hg_1, Hg_2, \ldots, Hg_n$ be the distinct right cosets of H in G and let $T_1, T_2, \ldots, T_k$ partition G into CSLCR of H in G. Further assume θ is a complete mapping of G.

The following layouts give orthogonal Cayley tables where the list $\theta[S]^{-1} := [\theta(s)^{-1} : s \in [S]]$.

	$[Hg_1]$	$[Hg_2]$	$\ldots$	$[Hg_n]$
$[T_1]$				
$[T_2]$				
$\vdots$				
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$[T_k]$				

	$[Hg_1]$	$[Hg_2]$	$\ldots$	$[Hg_n]$
$\theta[T_1]^{-1}$				
$\theta[T_2]^{-1}$				
$\vdots$				
$\theta[T_k]^{-1}$				

And they are orthogonal Cayley-Sudoku tables provided $\theta[T_1]^{-1}$, $\theta[T_2]^{-1}, \ldots, \theta[T_k]^{-1}$ still partition G into CSLCR of H in G.

Questions about Orthogonal Cayley-Sudoku Tables

- For any group having a complete mapping does there exist a subgroup H and a partition into CSLCR Cayley-Sudoku table for which the θ^{-1} Construction yields a pair of orthogonal Cayley-Sudoku tables based on H?
  (So far, yes.)

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  (So far, yes.)
- Is there a characterization (along the lines of Hall-Paige) of groups and subgroups for which orthogonal Cayley-Sudoku tables based on those subgroups exist?
  (G has orthogonal C-S tables based on H iff G has trivial or non-cyclic Sylow 2-subgroups and H is ????)

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- Is there a characterization (along the lines of Hall-Paige) of groups and subgroups for which orthogonal Cayley-Sudoku tables based on those subgroups exist?
  (G has orthogonal C-S tables based on H iff G has trivial or non-cyclic Sylow 2-subgroups and H is ????)
- How many mutually orthogonal Cayley-Sudoku tables can a group have?

Definition

*A **Magic Cayley-Sudoku table** is a Cayley-Sudoku table in which the blocks are magic squares, that is, the blocks are square and the group product (sum) of the elements in every row, column, and diagonal is the same group element, called the **magic constant**.*

$$\begin{array}{c|cccc|c} & & & & & \\ \hline & \star & \star & \cdots & \star & \rightarrow \\ & \star & \star & \cdots & \star & \rightarrow \\ & \vdots & \vdots & & \vdots & \vdots \\ & \star & \star & \cdots & \star & \rightarrow \\ \hline \swarrow & \downarrow & \downarrow & \cdots & \downarrow & \searrow \end{array}$$

All products indicated by the arrows in the indicated directions are the same.

Example

The $\mathbb{Z}_9$ Cayley-Sudoku table is (irreparably) not magic.

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	00	10	20	01	11	21	02	12	22
00	00	10	20	01	11	21	02	12	22
01	01	11	21	02	12	22	00	10	20
02	02	12	22	00	10	20	01	11	21
10	10	20	00	11	21	01	12	22	02
11	11	21	01	12	22	02	10	20	00
12	12	22	02	10	20	00	11	21	01
20	20	00	10	21	01	11	22	02	12
21	21	01	11	22	02	12	20	00	10
22	22	02	12	20	00	10	21	01	11

A Magic Cayley-Sudoku Table for $\mathbb{Z}_3 \times \mathbb{Z}_3$ with magic constant 00
($00 = (0,0)$, $10 = (1,0)$, etc.)

Specializations for Magic Cayley-Sudoku Tables

- Let $H = \{h_1, h_2, \ldots, h_k\}$ and let T be one CSLCR.
  Then $Th_1, Th_2, \ldots, Th_k$ partition G into CSLCR.

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- When H is a normal subgroup, we take $[T]$ as our right coset representatives for the columns.
- Our Magic Cayley-Sudoku tables are made this way.

	$[Ht_1]$	$[Ht_2]$	$\ldots$	$[Ht_n]$
$[Th_1]$				
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Magic Cayley-Sudoku Table Construction (RM & MW 2013)

- G is a group of order k^2 and $H \leq G$ of order k.

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- There exists a CSLCR $[T] = [t_1, t_2, \ldots, t_k]$ of H in G such that for every $t \in T$, $\prod_{j=1}^{k} (t_j t) = 1$ (**Magic Shuffle Property**).

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- Then the following layout gives a Magic Cayley-Sudoku Table with Magic Constant 1. (In fact, pandiagonal magic.)

	$[Ht_1]$	$[Ht_2]$	$\ldots$	$[Ht_n]$
$[Th_1]$				
$[Th_2]$				
$\vdots$				
$[Th_k]$				

The Main Diagonal Product in Typical Block

Write out a typical block

	$[Ht]$
$[Th]$	

	$h_1 t$	$h_2 t$	$\dots$	$h_k t$
$t_1 h$	$(t_1 h)(h_1 t)$			
$t_2 h$		$(t_2 h)(h_2 t)$		
$\vdots$			$\ddots$	
$t_k h$				$(t_k h)(h_k t)$

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$t_1 h$	$(t_1 h)(h_1 t)$			
$t_2 h$		$(t_2 h)(h_2 t)$		
$\vdots$			$\ddots$	
$t_k h$				$(t_k h)(h_k t)$

Multiply the main diagonal

$$\prod_{j=1}^{k}(t_j h)(h_j t) = h^k\left(\prod_{j=1}^{k} h_j\right)\left(\prod_{j=1}^{k}(t_j t)\right) = 1$$

using $h, h_j \in H \leq Z(G)$, $\exp(G)$ divides k, Trivial Product Property, & Magic Shuffle Property.

Observation

An abelian group has the Trivial Product Property iff it has trivial or non-cyclic Sylow 2-subgroups.

In the construction, $H \leq Z(G)$, so the Trivial Product Property is just another way of saying H has trivial or non-cyclic Sylow 2-subgroups.

Application: Abelian Groups

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- $\exp(G)$ divides k ~~and~~ $\cancel{H \leq Z(G)}$ (redundant)
- H has the Trivial Product Property.

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- ($\mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ show we cannot drop the exponent or Trivial Product Property conditions.)

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Bonus Slide: Orthogonal Magic Cayley-Sudoku Tables!

	00	12	21	11	20	02	22	01	10
00	00	12	21	11	20	02	22	01	10
20	20	02	11	01	10	22	12	21	00
10	10	22	01	21	00	12	02	11	20
12	12	21	00	20	02	11	01	10	22
02	02	11	20	10	22	01	21	00	12
22	22	01	10	00	12	21	11	20	02
21	21	00	12	02	11	20	10	22	01
11	11	20	02	22	01	10	00	12	21
01	01	10	22	12	21	00	20	02	11

	00	12	21	11	20	02	22	01	10
00	00	12	21	11	20	02	22	01	10
10	10	22	01	21	00	12	02	11	20
20	20	02	11	01	10	22	12	21	00
21	21	00	12	02	11	20	10	22	01
01	01	10	22	12	21	00	20	02	11
11	11	20	02	22	01	10	00	12	21
12	12	21	00	20	02	11	01	10	22
22	22	01	10	00	12	21	11	20	02
02	02	11	20	10	22	01	21	00	12

Addendum

Theorem (Vaughn-Lee & Wanless (2003))

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With the Hall-Paige conjecture, the above are equivalent to the admissibility of G, the existence of an orthogonal mate to the Cayley table of G.