Magic Cayley-Sudoku Tables

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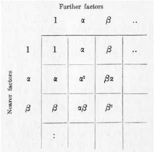

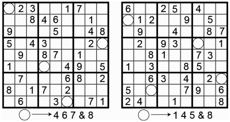

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

First Cayley Table (1854) & First Sudoku Puzzle (1979) & First Cayley-Sudoku Table (2010)

Definition

*A **Sudoku table** is an $n \times n$ array partitioned into rectangular blocks of some fixed size such that each of n symbols appear exactly once in each row, each column, and each block.*

In the standard Sudoku puzzle, the array is 9×9, the blocks are 3×3 and the symbols are the numbers 1 through 9.

9	3	6	1	4	7	8	2	5
1	4	7	2	5	8	9	3	6
8	2	5	9	3	6	7	1	4
3	6	9	4	7	1	2	5	8
4	7	1	5	8	2	3	6	9
2	5	8	3	6	9	1	4	7
6	9	3	7	1	4	5	8	2
7	1	4	8	2	5	6	9	3
5	8	2	6	9	3	4	7	1

A Standard Sudoku Table

Definition

*A **Cayley table** is an operation table for a finite group.*

Definition

*A **Cayley-Sudoku table** is a Cayley table which is also a (bordered) Sudoku table.*

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

A Cayley-Sudoku Table for $\mathbb{Z}_9$ (with 9=0)

Definition

A ***Magic Cayley-Sudoku table*** *is a Cayley-Sudoku table in which the blocks are magic squares, that is, the blocks are square and the group sum of the elements in every row, column, and diagonal is the same group element, called the* ***magic constant****.*

	$\star$	$\star$	$\cdots$	$\star$	$\longrightarrow$
	$\star$	$\star$	$\cdots$	$\star$	$\longrightarrow$
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$\star$	$\star$	$\cdots$	$\star$	$\longrightarrow$
$\swarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\downarrow$	$\searrow$

All sums indicated by the arrows are the same.

Non-example

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

NOT a Magic Cayley-Sudoku Table for $\mathbb{Z}_9$
(compare row sums in the first block)

Example

	00	10	20	01	11	21	02	12	22
00	00	10	20	01	11	21	02	12	22
01	01	11	21	02	12	22	00	10	20
02	02	12	22	00	10	20	01	11	21
10	10	20	00	11	21	01	12	22	02
11	11	21	01	12	22	02	10	20	00
12	12	22	02	10	20	00	11	21	01
20	20	00	10	21	01	11	22	02	12
21	21	01	11	22	02	12	20	00	10
22	22	02	12	20	00	10	21	01	11

A Magic Cayley-Sudoku Table for $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ with magic constant 00
($00 = (0,0)$, $10 = (1,0)$, etc.)

How was it made?

Take the subgroup $H := <10> = \{00, 10, 20\}$.

Take the complete set of coset representatives $T := \{00, 01, 02\}$

Arrange the table like this.

		$H+00$			$H+01$			$H+02$		
		00	10	20	01	11	21	02	12	22
	00	00	10	20	01	11	21	02	12	22
$T+00$	01	01	11	21	02	12	22	00	10	20
	02	02	12	22	00	10	20	01	11	21
	10	10	20	00	11	21	01	12	22	02
$T+10$	11	11	21	01	12	22	02	10	20	00
	12	12	22	02	10	20	00	11	21	01
	20	20	00	10	21	01	11	22	02	12
$T+20$	21	21	01	11	22	02	12	20	00	10
	22	22	02	12	20	00	10	21	01	11

That arrangement guarantees a Cayley-Sudoku Table.
(Carmichael, Schloeman & Ward, *Math. Mag.* April 2010, or Theorem 1.5.5 in J. Dénes and A. D. Keedwell, *Latin Squares and Their Applications*, 1974.)

What makes the magic?

- the sum of the elements in H is 00
- the sum of the elements of T is 00
- $3 \cdot x := x + x + x = 00$ for every $x \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$
  (the exponent of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is 3)

Typical Column Sum

		$\star$	g	$\star$	
			$\vdots$		
$00+h$		$\star$	$00+h+g$	$\star$	
$01+h$	$\cdots$	$\star$	$01+h+g$	$\star$	$\cdots$
$02+h$		$\star$	$02+h+g$	$\star$	
			$\vdots$		

column sum $= (00+01+02)+3\cdot h+3\cdot g = 00+00+00 = 00$
(using the sum of elements of T is 00 and exponent is 3)

Row sums use the exponent again and that the sum of elements of H is 00. Diagonal sums use everything.

Definition

A finite abelian (commutative) group is a ***zero-sum group*** *provided the sum of its elements is 0 (the group identity).*

(Aside: A finite abelian group is a zero-sum group if and only if it is has odd order or has a noncyclic Sylow 2-subgroup.)

Theorem

Suppose G is an abelian group of order n^2 and $H = \{h_1, h_2, \ldots, h_n\}$ is subgroup of G of order n. Assume $\exp(G)$ divides n; and H and G/H are zero-sum groups. Then there exists a set $T = \{t_1, t_2, \ldots, t_n\}$ of coset representatives of H in G whose sum is 0 (the group identity). Furthermore, the following layout produces a Magic Cayley-Sudoku Table with magic constant 0.

	$H + t_1$	$H + t_2$	$\cdots$	$H + t_n$
$T + h_1$				
$T + h_2$				
$\vdots$				
$T + h_n$				

Examples ($\mathbb{Z}_9$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, respectively) show the exponent and zero-sum conditions cannot be dropped.

Another (Half) Example

	00	20	02	22	10	30	12	32	01	21	03	23	33	13	31	11
00	00	20	02	22	10	30	12	32	01	21	03	23	33	13	31	11
10	10	30	12	32	20	00	22	02	11	31	13	33	03	23	01	21
01	01	21	03	23	11	31	13	33	02	22	00	20	30	10	32	12
33	33	13	31	11	13	23	01	21	30	10	32	12	22	02	20	00
20	20	00	22	12	30	10	32	12	21	01	23	03	13	33	11	31
30	30	10	32	12	00	20	02	22	31	11	13	33	23	03	21	01
21	21	01	23	03	31	11	33	13	22	02	20	00	10	30	12	32
13	13	33	11	31	23	03	21	01	10	30	12	32	02	22	00	20

and so forth …

Magic Cayley-Sudoku Table for $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ with magic constant 00

$$H = \langle 2 \rangle \oplus \langle 2 \rangle$$

$$T = \{00, 10, 01, 33\}$$

Open Questions

- Are there other constructions?
- What about non-abelian groups?
  (We have an example of order 81, but no theorem.)
- What is the minimum number of entries that determine a Magic Cayley-Sudoku Table?
- Can interesting puzzles be made?

Addendum – 23 January 2013

Zero sum groups have interesting connections. Consider these classes of finite (not necessarily abelian) groups.

$C_1 := (G:$ the sum of the elements of G *in some order* is 0)
$C_2 := (G: G$ has trivial or non-cyclic Sylow 2-subgroups)
$C_3 := (G:$ the Cayley Table of G has an orthogonal mate)
$C_4 := (G: G$ is admissible, i.e. admits a complete mapping)

We proved $C_1 = C_2$ in the abelian case. It is known that $C_3 = C_4$ ("well-known") and that $C_4 \subseteq C_1$ (Paige 1951). The famous Hall-Paige Conjecture is $C_2 \subseteq C_4$. Evidently, using the classification of finite simple groups, all these classes are now known to be equal.[1]

We wonder if equality of those classes might shed some light on how to generalize our construction to non-abelian groups.

[1] But take that with a grain of salt.