Cosets and Cayley-Sudoku Tables

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The wildly popular Sudoku puzzles [2] are 9×9 arrays divided into nine 3×3 sub-arrays or blocks. Digits 1 through 9 appear in some of the entries. Other entries are blank. The goal is to fill the blank entries with the digits 1 through 9 in such a way that each digit appears exactly once in each row and in each column, and in each block. Table 1 gives an example of a completed Sudoku puzzle.

One proves in introductory group theory that every element of any group appears exactly once in each row and once in each column of the group's operation or Cayley table. (In other words, any Cayley table is a Latin square.) Thus, every Cayley table has two-thirds of the properties of a Sudoku table; only the subdivision of the table into blocks that contain each element exactly once is in doubt. A question naturally leaps to mind: When and how can a Cayley table be arranged in such a way as to satisfy the additional requirements of being a Sudoku table? To be more specific, group elements labeling the rows and the columns of a Cayley table may be arranged in any order. Moreover, in defiance of convention, row labels and column labels need not be in the same order. Again we ask, when and how can the row and

9	3	6	1	4	7	2	5	8
1	4	7	2	5	8	3	6	9
2	5	8	3	6	9	4	7	1
3	6	9	4	7	1	5	8	2
4	7	1	5	8	2	6	9	3
5	8	2	6	9	3	7	1	4
6	9	3	7	1	4	8	2	5
7	1	4	8	2	5	9	3	6
8	2	5	9	3	6	1	4	7

Table 1: A Completed Sudoku Puzzle

column labels be arranged so that the Cayley table has blocks containing each group element exactly once?

For example, Table 2 shows that the completed Sudoku puzzle in Table 1 is actually a Cayley table of $\mathbb{Z}_9 := \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ under addition modulo 9. (We use 9 instead of the usual 0 in order to maintain the Sudoku-like appearance.)

As a second example, consider A_4, the alternating group on 4 symbols. We seek to arrange its elements as row and column labels so that the resulting Cayley table forms a Sudoku-like table, one in which the table is subdivided into blocks such that each group element appears exactly once in each block (as well as exactly once in each column and in each row, which, as noted, is always so in a Cayley table). Table 3 shows such an arrangement with 6×2 blocks. (In constructing the table, we operate with the row label on the left and column label on the right. Permutations are composed right to left. For example, the entry in row (14)(23), column (134) is (14)(23)(134) = (123).)

We say Tables 2 and 3 are Cayley-Sudoku tables of $\mathbb{Z}_9$ and A_4, respectively. In general, a *Cayley-Sudoku table* of a finite group G is a Cayley table for G subdivided into uniformly sized rectangular blocks in such a way that each group element appears exactly once in each block.

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

Table 2: A Cayley Table of $\mathbb{Z}_9$ with Sudoku Properties

	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124
(1)	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124
(13)(24)	(13)(24)	(14)(23)	(1)	(12)(34)	(142)	(134)	(123)	(243)	(234)	(124)	(132)	(143
(123)	(123)	(134)	(243)	(142)	(132)	(124)	(143)	(234)	(1)	(14)(23)	(12)(34)	(13)(2
(243)	(243)	(142)	(123)	(134)	(143)	(234)	(132)	(124)	(12)(34)	(13)(24)	(1)	(14)(2
(132)	(132)	(234)	(124)	(143)	(1)	(13)(24)	(14)(23)	(12)(34)	(123)	(142)	(134)	(243
(143)	(143)	(124)	(234)	(132)	(12)(34)	(14)(23)	(13)(24)	(1)	(243)	(134)	(142)	(123
(12)(34)	(12)(34)	(1)	(14)(23)	(13)(24)	(243)	(123)	(134)	(142)	(143)	(132)	(124)	(234
(14)(23)	(14)(23)	(13)(24)	(12)(34)	(1)	(134)	(142)	(243)	(123)	(124)	(234)	(143)	(132
(134)	(134)	(123)	(142)	(243)	(124)	(132)	(234)	(143)	(14)(23)	(1)	(13)(24)	(12)(3
(142)	(142)	(243)	(134)	(123)	(234)	(143)	(124)	(132)	(13)(24)	(12)(34)	(14)(23)	(1)
(234)	(234)	(132)	(143)	(124)	(13)(24)	(1)	(12)(34)	(14)(23)	(142)	(123)	(243)	(134
(124)	(124)	(143)	(132)	(234)	(14)(23)	(12)(34)	(1)	(13)(24)	(134)	(243)	(123)	(142

Table 3: A Cayley Table of A_4 with Sudoku Properties

Uninteresting Cayley-Sudoku tables can be made from any Cayley table of any group by simply defining the blocks to be the individual rows (or columns) of the table. Our goal in this note is to give three methods for producing interesting tables using cosets, thereby uncovering new applications of cosets. (See any introductory group theory text for a review of cosets, for example, [1, Chapter 7].)

Cosets Revisit Table 2. The cyclic subgroup generated by 3 in $\mathbb{Z}_9$ is $\langle 3\rangle = \{9, 3, 6\}$. The right and left cosets of $\langle 3\rangle$ in $\mathbb{Z}_9$ are $\langle 3\rangle+9 = \{9, 3, 6\} = 9+\langle 3\rangle$,

$\langle 3\rangle + 1 = \{1, 4, 7\} = 1 + \langle 3\rangle$, and $\langle 3\rangle + 2 = \{2, 5, 8\} = 2 + \langle 3\rangle$. With only a little prompting, we quickly see that the columns in each block are labeled by elements of right cosets of $\langle 3\rangle$ in $\mathbb{Z}_9$. Each set of elements labeling the rows of a block contains exactly one element from each left coset. Equivalently, the row labels partition $\mathbb{Z}_9$ into *complete sets of left coset representatives* of $\langle 3\rangle$ in $\mathbb{Z}_9$. (Momentarily we shall see why we bothered to distinguish between right and left.)

Reexamining Table 3 reveals a similar structure. Consider the subgroup $H := \langle(12)(34)\rangle = \{(1), (12)(34)\}$. We brush-up on composing permutations (right to left) by calculating the right coset $H(123) = \{(1)(123), (12)(34)(123)\} = \{(123), (243)\}$ and the corresponding left coset $\{(123)(1), (123)(12)(34)\} = \{(123), (134)\}$. In that fashion, we find the right cosets to be $H(1) = \{(1), (12)(34)\}$, $H(13)(24) = \{(13)(24), (14)(23)\}$, $H(123) = \{(123), (243)\}$, $H(142) = \{(142), (134)\}$, $H(132) = \{(132), (143)\}$, and $H(234) = \{(234), (124)\}$ while the left cosets are $(1)H = \{(1), (12)(34)\}$, $(13)(24)H = \{(13)(24), (14)(23)\}$, $(123)H = \{(123), (134)\}$, $(243)H = \{(243), (142)\}$, $(132)H = \{(132), (234)\}$, $(143)H = \{(143), (124)\}$. This time we know what to expect. Sure enough, the columns in Table 3 are labeled by the elements of the distinct *right* cosets of H in A_4 while the row labels partition A_4 into complete sets of *left* coset representatives of H in A_4.

Those examples illustrate our first general construction.

Before proceeding, let us agree upon a convention for labeling a Cayley table. When a set is listed in a row or column of the table, it is to be interpreted as the individual elements of that set being listed in separate rows or columns, respectively. For example, under that convention, the rows and columns of Table 2 could be labeled

	$\{9, 3, 6\}$	$\{1, 4, 7\}$	$\ldots$
$\{9, 1, 2\}$			
$\{3, 4, 5\}$			
$\vdots$			

where the label $\{9, 1, 2\}$ is interpreted as the elements 9, 1, and 2 listed vertically, one per row, and $\{9,3,6\}$ is interpreted as the elements 9, 3, and 6 listed horizontally, one per column.

Cayley-Sudoku Construction 1 *Let G be a finite group. Assume H is a subgroup of G having order k and index n (so that $|G| = nk$). If $Hg_1, Hg_2, \ldots, Hg_n$ are the n distinct right cosets of H in G, then arranging the Cayley table of G with columns labeled by the cosets $Hg_1, Hg_2, \ldots, Hg_n$ and the rows labeled by sets $T_1, T_2, \ldots, T_k$ (as in Table 4) yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives of H in G.*

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

Table 4: Construction 1 Using Right Cosets and Left Coset Representatives

Furthermore, if $y_1H, y_2H, \ldots, y_nH$ are the n distinct left cosets of H in G, then arranging the Cayley table of G with rows labeled by the cosets $y_1H, y_2H, \ldots, y_nH$ and the columns labeled by sets $R_1, R_2, \ldots, R_k$ yields a Cayley-Sudoku table of G with blocks of dimension $k \times n$ if and only if $R_1, R_2, \ldots, R_k$ partition G into complete sets of right coset representatives of H in G.

Note that the second version of the Construction 1 is dual to the first, obtained by reversing the left with right and rows with columns.

Now let us prove the correctness of the construction for the case of right cosets. An arbitrary block of the table, indexed by $T_h = \{t_1, t_2, \ldots, t_n\}$ and Hg_i, is the given the following table.

	Hg_i
t_1	t_1Hg_i
t_2	t_2Hg_i
$\vdots$	$\vdots$
t_n	t_nHg_i

The elements in the block are the elements of the set $B := t_1Hg_i \cup t_2Hg_i \cup \ldots \cup t_nHg_i = (t_1H \cup t_2H \cup \ldots \cup t_nH)g_i$, the equality being a routine exercise. We want to show that the elements of G appear exactly once in that block if and only if T_h is a complete set of left coset representatives of H in G.

If T_h is a complete set of left coset representatives, then $t_1H \cup t_2H \cup \ldots \cup t_nH = G$. So we have $B = Gg_i = G$. Thus every element of G appears in every block. But the number of entries in the block is nk and the order of G is nk, so every element of G appears exactly once in each block.

On the other hand, if every element appears exactly once in each block, then $B = G$ and that gives us $G = Gg_i^{-1} = Bg_i^{-1} = t_1H \cup t_2H \cup \ldots \cup t_nH$. There are n cosets in this union, each having order k. Therefore, since the union of those cosets is the entire group G and $|G| = nk$, we must have n distinct cosets in the union. Thus, $\{t_1, t_2, \ldots, t_n\}$ is a complete set of left coset representatives of H in G, as claimed.

The first punch line of this section is that any proper non-trivial subgroup of a finite group gives rise to interesting Cayley-Sudoku tables by using Construction 1. Therefore, any finite group having a proper non-trivial subgroup (which is to say, any finite group whose order is not a prime) admits an interesting Cayley-Sudoku table.

We now introduce another construction utilizing cosets. This construction is "like-handed," that is, it uses left cosets and left coset representatives (or right and right). Construction 1 is "cross-handed." It uses right cosets and left coset representatives or left cosets and right coset representatives.

For this construction, we recall a standard group theoretic definition. For any subgroup H of a group G and for any $g \in G$, $g^{-1}Hg := \{g^{-1}hg : h \in H\}$ is called a *conjugate* of H and is denoted H^g. It is routine to show H^g is a subgroup of G.

Cayley-Sudoku Construction 2 *Assume H is a subgroup of G having order k and index n. Also suppose $t_1H,\ t_2H,\ \dots,\ t_nH$ are the distinct left cosets of H in G. Arranging the Cayley table of G with columns labeled by the cosets $t_1H,\ t_2H,\ \dots,\ t_nH$ and the rows labeled by sets $L_1, L_2, \dots, L_k$ yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $L_1, L_2, \dots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$.*

	t_1H	t_2H	$\dots$	t_nH
L_1				
L_2				
$\vdots$				
L_k				

Table 5: Construction 2 Using Left Cosets and Left Coset Representatives

Since one or two interesting subtleties arise, we will verify the correctness of Construction 2. Consider an arbitrary block in the table, indexed by $L_i := \{g_{i1}, g_{i2}, \dots g_{in}\}$ and t_jH. Note, however, that $t_jH = t_jHt_j^{-1}t_j = H^{t_j^{-1}}t_j$. Thus, our block is the following.

	$H^{t_j^{-1}}t_j$
g_{i1}	$g_{i1}H^{t_j^{-1}}t_j$
g_{i2}	$g_{i2}H^{t_j^{-1}}t_j$
$\vdots$	$\vdots$
g_{in}	$g_{in}H^{t_j^{-1}}t_j$

The set of elements in the block is $g_{i1}H^{t_j^{-1}}t_j \cup g_{i2}H^{t_j^{-1}}t_j \cup \dots \cup g_{1n}H^{t_j^{-1}}t_j = (g_{i1}H^{t_j^{-1}} \cup \dots \cup g_{in}H^{t_j^{-1}})t_j$.

Suppose $L_1, L_2, \ldots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$, then $L_i = \{g_{i1}, g_{i2}, \ldots, g_{in}\}$ is a complete set of left coset representatives of $H^{t_j^{-1}}$. Therefore, just as in the verification of Construction 1, we can show every element of G appears exactly once in the block.

Conversely, suppose every element of G appears exactly once in each block. Once again arguing as in Construction 1, we conclude each L_i is a complete set of left coset representatives of H^{t_j} for every j.

In order finish, we need a (known) result of independent group theoretic interest. Namely, with notation as in the Construction, for every $g \in G$, there exists t_j such that $H^g = H^{t_j^{-1}}$. To see that, let $g \in G$, then g^{-1} is in some left coset of H, say t_jH. Thus, $g^{-1} = t_jh$ for some $h \in H$. Armed with the observation $hHh^{-1} = H$ (easily shown since H is a subgroup), we hit our target: $H^g = g^{-1}Hg = (t_jh)H(h^{-1}t_j^{-1}) = t_j(hHh^{-1})t^{-1} = t_jHt_j^{-1} = H^{t_j^{-1}}$. Combining this with the preceding paragraph, we can conclude $L_1, L_2, \ldots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$, as claimed.

We invite the reader to formulate and verify a right-handed version of Construction 2. We also raise an interesting and, evidently, non-trivial question for further investigation. Under what circumstances can one decompose a finite group G in the way required by Construction 2?

There is one easy circumstance. If H is a normal subgroup of a G, then it is not difficult to show $H^g = H$ for every $g \in G$ [1, Chapter 9]. Thus, decomposing G into complete sets of left coset representatives of H will do the trick. Sadly, in that case, Construction 2 gives the same Cayley-Sudoku table as Construction 1 because the left cosets indexing the columns equal the corresponding right cosets by normality.

Happily, we know of one general circumstance in which we can decompose G in the desired way to obtain new Cayley-Sudoku tables. It is contained in the following proposition, stated without proof.

Proposition *Suppose the finite group G contains subgroups $T := \{t_1, t_2, \ldots, t_n\}$ and $H := \{h_1, h_2, \ldots, h_k\}$ such that $G = \{th : t \in T, h \in H\} := TH$ and $T \cap H = \{e\}$, then the elements of T form a complete set of left coset repre-*

sentatives of H and the cosets $Th_1, Th_2, \ldots, Th_k$ decompose G into complete sets of left coset representatives of H^g for every $g \in G$.

In other words, from the Proposition, Construction 2 applies when we set $L_i := Th_i$ and use the left cosets $t_1H, t_2H, \ldots, t_nH$. Let us try it out on the group S_4.

Let $H = \langle(123)\rangle = \{(1), (123), (132)\}$ and $T = \{(1), (12)(34), (13)(24), (14)(23), (24), (1234),$ One can check (by brute force, if necessary) that H and T are subgroups of S_4 satisfying the hypotheses of the Proposition. Therefore, according to Construction 2, the following table yields a Cayley-Sudoku table of S_4.

	H	$(12)(34)H$	$(13)(24)H$	$(14)(23)H$	$(24)H$	$(1234)H$	$(1432)H$	$(13)H$
T								
$T(123)$								
$T(132)$								

Seeking to be convinced that is a new Cayley-Sudoku table, not of the kind produced by Construction 1, we examine the sets indexing the columns and rows. In Construction 1, the sets indexing columns are right cosets of some subgroup or else the sets indexing the rows are left cosets of some subgroup. In our table, the only subgroup indexing the columns is H and most of the remaining index sets are not right cosets of H. For example, $(12)(34)H \neq H(12)(34)$ and so it is not a right coset of H. Similar consideration of the sets indexing the rows shows Construction 1 was not on the job here.

Extending Cayley-Sudoku tables Our final construction shows a way to extend a Cayley-Sudoku table of a subgroup to a Cayley-Sudoku table of the full group.

Cayley-Sudoku Construction 3 *Let G be a finite group with a subgroup A. Let $C_1, C_2, \ldots, C_k$ partition A and $R_1, R_2, \ldots R_n$ partition A such that the following table is a Cayley-Sudoku table of A.*

	C_1	C_2	$\ldots$	C_k
R_1				
R_2				
$\vdots$				
R_n				

If $\{l_1, l_2, \ldots, l_t\}$ and $\{r_1, r_2, \ldots r_t\}$ are complete sets of left and right coset representatives, respectively, of A in G, then arranging the Cayley table of G with columns labeled with the sets $C_i r_j$, $i = 1, \ldots, k$, $j = 1, \ldots, t$ and the b^{th} block of rows labeled with $l_j R_b$, $j = 1, \ldots, t$, for $b = 1, \ldots, n$ (as in Table 6) yields a Cayley-Sudoku table of G with blocks of dimension $tk \times n$.

	C_1r_1	C_2r_1	$\ldots$	C_kr_1	C_1r_2	$\ldots$	C_kr_2	$\ldots$	C_1r_t	$\ldots$	C_kr_t
l_1R_1 l_2R_1 $\vdots$ l_tR_1											
l_1R_2 $\vdots$ l_tR_2											
$\vdots$											
l_1R_n $\vdots$ l_tR_n											

Table 6: Construction 3

Proving the correctness of Construction 3 is quite like the proof for Construction 1. We leave it as an exercise for the reader and proceed to an example. Working in the group $\mathbb{Z}_8 := \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition modulo 8, let us apply Construction 1 to form a Cayley-Sudoku table for the subgroup $\langle 2 \rangle$ and then extend that table to a Cayley-Sudoku table of $\mathbb{Z}_8$ via Construction 3.

Observe that $\langle 4 \rangle = \{0, 4\}$ is a subgroup of $\langle 2 \rangle = \{0, 2, 4, 6\}$. The left and right cosets of $\langle 4 \rangle$ in $\langle 2 \rangle$ are $0 + \langle 4 \rangle = \{0, 4\} = \langle 4 \rangle + 0$ and $2 + \langle 4 \rangle = \{2, 6\} = \langle 4 \rangle + 0$. Thus, $\{0, 2\}$ and $\{4, 6\}$ partition $\mathbb{Z}_8$ into complete sets of right coset representatives. Applying Construction 1, wherein elements of left cosets label the rows and right coset representatives label the columns, yields Table 7.

	0	2	4	6
0	0	2	4	6
4	4	6	0	2
2	2	4	6	0
6	6	0	2	4

Table 7: Construction 1 Applied

Now the left and right cosets of $\langle 2 \rangle$ in $\mathbb{Z}_8$ are $0+\langle 2 \rangle = \{0, 2, 4, 6\} = \langle 2 \rangle+0$ and $1 + \langle 2 \rangle = \{1, 3, 5, 7\} = \langle 2 \rangle + 1$. Accordingly, $\{0, 1\}$ is a complete set of left and right coset representatives of $\langle 2 \rangle$ in $\mathbb{Z}_8$. According to Construction 3, Table 8 should be (and is, much to our relief) a Cayley-Sudoku table of $\mathbb{Z}_8$. For easy comparison with Table 6, rows and columns are labeled both with sets and with individual elements.

We chose $\mathbb{Z}_8$ for our example because it has enough subgroups to make Construction 3 interesting, yet the calculations are easy to do and the resulting table fits easily on a page. In one sense, however, the calculations are too easy. Since $\mathbb{Z}_8$ is abelian, all the corresponding right and left cosets of any subgroup are equal. (In other words, all the subgroups are normal.) Thus, the role of right versus left in Construction 3 is obscured. The interested reader may wish to work out an example where right and left cosets are different. For instance, in S_4, one could consider the subgroup $A := \{(1), (12)(34), (13)(24), (14)(23), (24), (1234), (1432), (13)\}$. Use Con-

		$\{0,2\}+0$		$\{0,2\}+1$		$\{4,6\}+0$		$\{4,6\}+0$	
		0	2	1	3	4	6	5	7
$0+\{0,4\}$	0	0	2	1	3	4	6	5	7
	4	4	6	5	7	0	2	1	3
$1+\{0,4\}$	1	1	3	2	4	5	7	6	0
	5	5	7	6	0	1	3	2	4
$0+\{2,6\}$	2	2	4	3	5	6	0	7	1
	6	6	0	7	1	2	4	3	5
$1+\{2,6\}$	3	3	5	4	6	7	1	0	2
	7	7	1	0	2	3	5	4	6

Table 8: A Cayley-Sudoku Table of $\mathbb{Z}_8$ from Construction 3

struction 1 with the subgroup $\langle(24)\rangle$ of A to obtain a Cayley-Sudoku table of A, then apply Construction 3 to extend that table to a Cayley-Sudoku table of S_4. The associated computations are manageable (barely, one might think by the end!) and the roles of right and left are more readily apparent.

Table 8 is a new sort of Cayley-Sudoku table, one not produced by either of Constructions 1 or 2. To see why, recall that in Construction 1 and 2 (including the right-handed cousin of 2), either the columns or the rows in the blocks are labeled by cosets of a subgroup. One of those cosets is, of course, the subgroup itself. However, we easily check that none of the sets labeling columns or rows of the blocks in Table 8 are subgroups of $\mathbb{Z}_8$.

A puzzle The ubiquitous Sudoku leads many students to treat the familiar exercise of filling in the missing entries of a partial Cayley table as a special sort of Sudoku puzzle. In a recent group theory course taught by the third author, several students explained how they deduced missing entries in such an exercise [1, exercise 25 p. 55] by writing "I Sudokued them." Meaning they applied Sudoku-type logic based on the fact that rows and columns of a Cayley table contain no repeated entries.

We extend that notion by including a Cayley-Sudoku puzzle for the reader. It requires both group theoretic and Sudoku reasoning. The group theory required is very elementary. (In particular, one need not use the classification of groups of order 8.)

The puzzle has three parts, one for entertainment and two to show this is truly a new sort of puzzle. First, complete Table 9 with 2×4 blocks as indicated so that it becomes a Cayley-Sudoku table. Do not assume *a priori* that Table 9 was produced by any of Constructions 1-3. Second, show group theoretic reasoning is actually needed in the puzzle by completing Table 9 so that it satisfies the three Sudoku properties for the indicated 2×4 blocks but is not the Cayley table of any group. Third, show Sudoku reasoning is required by finding another way to complete Table 9 so that it is a Cayley table of some group, but not a Cayley-Sudoku table.

	1	2	3	4	5	6	7	8
1							7	
5					1			
2				1				
6						1		
3					7			
7				6			1	
4								
8				7				

Table 9: A Cayley-Sudoku Puzzle

Ideas for further study By exhaustive (in more ways than one) analysis of cases, the authors can show that the only 9×9 Cayley-Sudoku tables are those resulting from Construction 1. Is the same true for $p^2 \times p^2$ Cayley-Sudoku tables where p is a prime?

All the constructions of Cayley-Sudoku tables known to the authors, including some not presented in this paper, ultimately rely on cosets and coset representatives. Are there Cayley-Sudoku constructions that do not use cosets and coset representatives?

Related to the previous question, how does one create a single block of a Cayley-Sudoku table? That is, if G is a group with subsets (not necessarily subgroups) K and H such that $|G| = |K||H|$, what are "nice" conditions under which we will have $KH = G$?

Can a Cayley-Sudoku table of a group be used to construct a Cayley-Sudoku table of a subgroup or a factor group? Can a Cayley-Sudoku table of a factor group be used to construct a Cayley-Sudoku table of the original group?

Are there efficient algorithms for generating interesting Cayley-Sudoku puzzles?

Making the definition of Cayley-Sudoku tables less restrictive can lead to some interesting examples. For instance if the definition of Cayley-Sudoku tables is altered so that the individual blocks of the table do not have to be of fixed dimension we obtain in Table 10 an example of a *generalized Cayley-Sudoku* table of the group $\mathbb{Z}_8$.

What is a construction method for such generalized Cayley-Sudoku tables? How about for *jigsaw Cayley-Sudoku tables* wherein the blocks are not rectangles?

Perhaps most interesting of all, find other circumstances under which Construction 2 applies.

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	0	4	1	3	5	7	2	6
0	0	4	1	3	5	7	2	6
1	1	5	2	4	6	0	3	7
2	2	6	3	5	7	1	4	0
3	3	7	4	6	0	2	5	1
4	4	0	5	7	1	3	6	2
5	5	1	6	0	2	4	7	3
6	6	2	7	1	3	5	0	4
7	7	3	0	2	4	6	1	5

Table 10: A Generalized Cayley-Sudoku Table

thank one another for many hours of satisfying and somewhat whimsical mathematics. We also thank the students at DigiPen Institute of Technology, where the first author was a guest speaker, for suggesting that we make a Cayley-Sudoku puzzle.

References

1. J.A. Gallian, *Contemporary Abstract Algebra*, 6th ed., Houghton Mifflin Co., New York, NY, 2006.

2. R. Wilson, The Sudoku Epidemic, *FOCUS* **26** (2006), 5-7.

Puzzle Solutions In each solution, the original puzzle entries are in bold face for easy identification.

Part 1. Table 11 shows the solution. Clearly, it satisfies the three Sudoku properties. We will show it is the Cayley table of D_4, the dihedral group of order 8. We will regard D_4 as the group of symmetries of a square. Let R_{90} be a counterclockwise rotation about the center of the square and let H be a reflection across a line through the center of the square that is parallel to a side of the square. The eight elements of D_4 are R_{90}^0, R_{90}^1, R_{90}^2, R_{90}^3, HR_{90}^0,

HR_{90}^1, HR_{90}^2, and HR_{90}^3. Numbering those elements 1 through 8 in the order given and then calculating the Cayley table gives Table 11. Thus, we have a Cayley-Sudoku table as claimed. By the way, it was obtained by applying Construction 1 to the subgroup $\langle R_{90} \rangle$.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	***7***	8
5	5	6	7	8	***1***	2	3	4
2	2	3	4	***1***	8	5	6	7
6	6	7	8	5	4	***1***	2	3
3	3	4	1	2	***7***	8	5	6
7	7	8	5	***6***	3	4	***1***	2
4	4	1	2	3	6	7	8	5
8	8	5	6	***7***	2	3	4	1

Table 11: Cayley-Sudoku Puzzle Solution

Part 2. Table 12 visibly satisfies the Sudoku conditions. It is not a Cayley table. For otherwise, $1 \cdot 7 = 7$ implies 1 is the identity. However, $1 \cdot 2 \neq 2$.

	1	2	3	4	5	6	7	8
1	1	3	2	4	5	6	***7***	8
5	5	7	6	8	***1***	2	3	4
2	2	4	3	***1***	8	5	6	7
6	6	8	7	5	4	***1***	2	3
3	3	1	4	2	***7***	8	5	6
7	7	5	8	***6***	3	4	***1***	2
4	4	2	1	3	6	7	8	5
8	8	6	5	***7***	2	3	4	1

Table 12: Sudoku-not-Cayley Puzzle Solution

Part 3. Table 13 does not satisfy the Sudoku conditions. Blocks contain repeated entries. It is, however, a Cayley table. One can check that it is again the Cayley table of D_4. Just change the labeling of R_{90}^2 from 3 to 5 and of H from 5 to 3. Table 13 is the recalculated Cayley table.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	***7***	8
5	5	4	7	2	***1***	8	3	6
2	2	5	8	***1***	4	3	6	7
6	6	7	4	3	8	***1***	2	5
3	3	6	1	8	***7***	2	5	4
7	7	8	5	***6***	3	4	***1***	2
4	4	1	6	5	2	7	8	3
8	8	3	2	***7***	6	5	4	1

Table 13: Cayley-not-Sudoku Puzzle Solution