Cayley-Sudoku Tables: A WOU (Re)Discovery

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Pi Mu Epsilon Oregon Delta Chapter Induction Ceremony
Pi Day 2016

Outline

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Groups

A *group* is a set with an operation. The operation must be closed and associative. There must be an identity. Each element must have an inverse.

In this talk, all groups are finite, meaning the set has only finitely many elements.

An Example of a Group

Set: $\mathbb{Z}_9 := \{1,2,3,4,5,6,7,8,9\}$
(Think of 9 as a badly written 0.)

Operation: Addition mod 9, denoted $+_9$

- For every $x, y \in \mathbb{Z}_9$,
  $x +_9 y := x + y \bmod 9 := \operatorname{mod}(x + y, 9)$

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- For kids, it's "clock arithmetic" on a clock with 9 hours.
- Examples:
  - $3 +_9 8 := 3 + 8 \text{ mod } 9 := \text{ mod}\,(3 + 8, 9) = 2$
  - $3 +_9 6 = 9$
- Closure is clear. 9 is the identity. Inverses are easy to spot.
  Trust me on associativity. $\therefore$ It is a group.

Arthur Cayley 1821-1895

Distinguished student at Cambridge. Graduated 1842. Barrister in London 1849-1863. Sadleirian Professor of Pure Mathematics at Cambridge 1863. Collected works in 13 volumes contain over 900 papers, including …

The First Paper on Abstract Group Theory

ON THE THEORY OF GROUPS, AS DEPENDING UPON THE SYMBOLIC EQUATION $\theta^n = 1$

Arthur Cayley

Philosophical Magazine, VOL. VII (1854)

After defining a group (approximately as we do now), Cayley writes,

> It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table, thus:
>
> Further factors
>
> | | | 1 | α | β | .. |
> |---|---|---|---|---|---|
> | Nearer factors | 1 | 1 | α | β | .. |
> | | α | α | α^2 | $\beta\alpha$ | |
> | | β | β | $\alpha\beta$ | β^2 | |
> | | | : | | | |
>
> that as well each line as each column of the square will contain all the symbols $1,\ \alpha,\ \beta, \ldots$.

- The table thus described by Cayley is now called the *Cayley Table* of the group.

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- Cayley claims that it has 2/3 of the properties of a Sudoku-like table, that is, each symbol occurs (exactly) once in each row and exactly once in each column. Such a table is called a *Latin Square.*

(Unorthodox) Cayley Table of $\mathbb{Z}_9$ with operation $+_9$

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

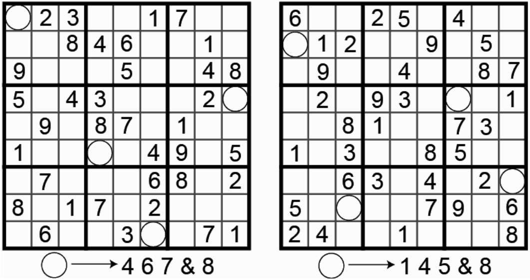
Hold that thought …

Whence Sudoku?

According to Ed Pegg, Jr. (MAA website),

> *In the May 1979 issue of Dell Pencil Puzzles & Word Games (issue #16), page 6, something amazing appeared:* ***Number Place****. Here are the original instructions: "In this puzzle, your job is to place a number into every empty box so that each row across, each column down, and each small 9-box square within the large square (there are 9 of these) will contain each number from 1 through 9. Remember that no number may appear more than once in any row across, any column down, or within any small 9-box square; this will help you solve the puzzle …*

. . . The numbers in circles below the diagram will give you a head start–each of these four numbers goes into one of the circle boxes in the diagram (not necessarily in the order given)."



The first Number Place puzzles. (*Dell Pencil Puzzles & Word Games* #16, page 6, 1979-05)

- Pegg cites personal communication with Will Shortz (*NY Times* crossword puzzle editor, NPR Puzzlemaster, and “star” of the movie *Wordplay*), who found the puzzle was invented by 74 year old architect Howard Garns (1905-1989).

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- The speaker first saw a Sudoku puzzle in the possession of Professor Sam Hall, Willamette U, July 2005.

Drum roll, please.

Divide the Cayley table of $\mathbb{Z}_9$ into nine 3 by 3 blocks, like a Sudoku puzzle.

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

Tah-dah! The First Cayley-Sudoku Table (under that name)

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

Mathematics Magazine, April 2010

It is a Cayley table of $\mathbb{Z}_9$ and it is also a Sudoku table because it is divided into blocks in which each group element appears exactly once.

Cayley-Sudoku Puzzles

Given a partially completed Cayley-Sudoku Table of an unknown group, complete the table so that each group element appears exactly once in each row, in each column, and in each designated block.

Hints

- The usual Sudoku techniques.

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- Look for the identity.

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- If you find $x \star y =$ identity, then you also know $y \star x =$ identity.

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Hints

- The usual Sudoku techniques.
- Look for the identity.
- If you find $x \star y =$ identity, then you also know $y \star x =$ identity.
- In the given puzzle, the group is not $\mathbb{Z}_8$. The puzzle can be done without knowing the actual group.

Cayley-Sudoku Puzzle with 2×4 blocks

	1	2	3	4	5	6	7	8
1							7	
5					1			
2				1				
6						1		
3					7			
7				6			1	
4								
8				7				

Cayley-Sudoku Puzzle Solution

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	***7***	8
5	5	6	7	8	***1***	2	3	4
2	2	3	4	***1***	8	5	6	7
6	6	7	8	5	4	***1***	2	3
3	3	4	1	2	***7***	8	5	6
7	7	8	5	***6***	3	4	***1***	2
4	4	1	2	3	6	7	8	5
8	8	5	6	***7***	2	3	4	1

Constructions for Cayley-Sudoku Tables

With Jenn Carmichael ’06 and Keith Schloeman ’07

Every Cayley table has two of the three of the properties of a Sudoku table; only the subdivision of the table into blocks that contain each element exactly once is in doubt. When and how can a Cayley table be arranged in such a way as to satisfy the additional requirements of being a Sudoku table?

Examine our Cayley-Sudoku table of $\mathbb{Z}_9$ for clues.

Column Labels

	9	**3**	**6**	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

The set of the first three column labels $\{9,3,6\}$ is also a group under $+_9$. That makes it a *subgroup* of $\mathbb{Z}_9$.

Column Labels

	9	3	6	**1**	**4**	**7**	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

Add 1 to each of the elements of the subgroup: $9 +_9 1 = 1$, $3 +_9 1 = 4$, $6 +_9 1 = 7$, those are the next three column labels. The resulting set is called a *right coset* of the subgroup, it is denoted $\{9, 3, 6\} +_9 1$.

Column Labels

	9	3	6	1	4	7	**2**	**5**	**8**
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
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Now consider the right coset $\{9,3,6\} +_9 2 = \{9 +_9 2, 3 +_9 2, 6 +_9 2\} = \{2,5,8\}$. The elements of that coset are the final three column labels.

Column Labels

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9	9	3	6	1	4	7	2	5	8
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6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
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Observation 1: The columns in each block of the Cayley-Sudoku table are labeled with elements of the right cosets of a subgroup.

Row Labels

Left cosets of the subgroup are also of interest.

$9 +_9 \{9,3,6\} = \{9 +_9 9, 9 +_9 3, 9 +_9 6\} = \{9,3,6\}$

$1 +_9 \{9,3,6\} = \{1 +_9 9, 1 +_9 3, 1 +_9 6\} = \{1,4,7\}$

$2 +_9 \{9,3,6\} = \{2 +_9 9, 2 +_9 3, 2 +_9 6\} = \{2,5,8\}$

Notice that left and right cosets partition the group into disjoint subsets.

Row Labels

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
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$$9 +_9 \{9,3,6\} = \{9 +_9 9, 9 +_9 3, 9 +_9 6\} = \{9,3,6\}$$

$$1 +_9 \{9,3,6\} = \{1 +_9 9, 1 +_9 3, 1 +_9 6\} = \{1,4,7\}$$

$$2 +_9 \{9,3,6\} = \{2 +_9 9, 2 +_9 3, 2 +_9 6\} = \{2,5,8\}$$

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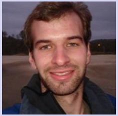
Observation 2: The rows in each block of the Cayley-Sudoku table are each labeled with a *complete set of left coset representatives*, that is, a *left transversal.*

Keith’s Construction



Let G with operation $\star$ be a finite group.
Take any subgroup S of G.
Arrange the Cayley table of G like this:
Columns labeled by the distinct right cosets $S \star g_1, S \star g_2, \ldots, S \star g_n$
Rows labeled by sets $T_1, T_2, \ldots, T_k$ where $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives of S in G.

	$S \star g_1$	$S \star g_2$	$\ldots$	$S \star g_n$
T_1				
T_2				
$\vdots$				
T_k				

This *always* gives a Cayley-Sudoku Table.

Another Example of a Group

D_4 = the set of symmetries of a square under the operation of composition of functions.

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- Rotations about the center (counterclockwise): $R_0, R_{90}, R_{180}, R_{270}$

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Eight Symmetries

- Rotations about the center (counterclockwise): $R_0, R_{90}, R_{180}, R_{270}$
- Reflections across lines through the center: H (horizontal), V (vertical), D and F (diagonal)

Right cosets of the subgroup $\{R_0, H\}$ will label the columns.

1. $\{R_0, H\} \circ R_0 := \{R_0 \circ R_0, H \circ R_0\} = \{R_0, H\}$
2. $\{R_0, H\} \circ R_{90} := \{R_0 \circ R_{90}, H \circ R_{90}\} = \{R_{90}, D\}$
3. $\{R_0, H\} \circ R_{180} := \{R_0 \circ R_{180}, H \circ R_{180}\} = \{R_{180}, V\}$
4. $\{R_0, H\} \circ R_{270} := \{R_0 \circ R_{270}, H \circ R_{270}\} = \{R_{270}, F\}$

Complete sets of left coset representatives of $\{R_0, H\}$ will label the rows.

1. $R_0 \circ \{R_0, H\} := \{R_0 \circ R_0, R_0 \circ H\} = \{R_0, H\}$
2. $R_{90} \circ \{R_0, H\} := \{R_{90} \circ R_0, R_{90} \circ H\} = \{R_{90}, F\}$
3. $R_{180} \circ \{R_0, H\} := \{R_{180} \circ R_0, R_{180} \circ H\} = \{R_{180}, V\}$
4. $R_{270} \circ \{R_0, H\} := \{R_{270} \circ R_0, R_{270} \circ H\} = \{R_{270}, D\}$

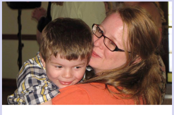
These sets do the trick:
$T_1 := \{R_0, R_{90}, V, D\}$ and $T_2 := \{H, F, R_{180}, R_{270}\}$

(Notice the left and right cosets are not the same.)

Keith's Construction Applied to D_4

	R_0	H	R_{90}	D	R_{180}	V	R_{270}	F
R_0	R_0	H	R_{90}	D	R_{180}	V	R_{270}	F
R_{90}	R_{90}	F	R_{180}	H	R_{270}	D	R_0	V
V	V	R_{180}	F	R_{270}	H	R_0	D	R_{90}
D	D	R_{270}	V	R_0	F	R_{90}	H	R_{180}
H	H	R_0	D	R_{90}	V	R_{180}	F	R_{270}
F	F	R_{90}	H	R_{180}	D	R_{270}	V	R_0
R_{180}	R_{180}	V	R_{270}	F	R_0	H	R_{90}	D
R_{270}	R_{270}	D	R_0	V	R_{90}	F	R_{180}	H

Jen & Mike's "Christmas Eve" Construction



Label the columns with *left* cosets, instead of right cosets.

	$t_1 \star S$	$t_2 \star S$	$\dots$	$t_n \star S$
L_1				
L_2				
$\vdots$				
L_k				

In order for the above to be a Cayley-Sudoku table, the sets $L_1, L_2, \dots, L_k$ labeling the rows must be complete sets of left coset representatives for S and (usually) several other subgroups at once!

Namely, for the subgroups $g^{-1} \star S \star g$ for all $g \in G$, where $g^{-1} \star S \star g := \{g^{-1} \star h \star g : h \in S\}$, which are the *conjugates* of S.

Use the subgroup $\{R_0, H\}$.

Left cosets of $\{R_0, H\}$ will label the columns.

1. $R_0 \circ \{R_0, H\} := \{R_0 \circ R_0, R_0 \circ H\} = \{R_0, H\}$
2. $R_{90} \circ \{R_0, H\} := \{R_{90} \circ R_0, R_{90} \circ H\} = \{R_{90}, F\}$
3. $R_{180} \circ \{R_0, H\} := \{R_{180} \circ R_0, R_{180} \circ H\} = \{R_{180}, V\}$
4. $R_{270} \circ \{R_0, H\} := \{R_{270} \circ R_0, R_{270} \circ H\} = \{R_{270}, D\}$

Rows must be labeled with complete sets of left coset representatives for $\{R_0, H\}$ and for the subgroup $\{R_0, V\} = R_{90}^{-1} \circ \{R_0, H\} \circ R_{90}$ (the only conjugates of $\{R_0, H\}$).

1. $R_0 \circ \{R_0, V\} := \{R_0 \circ R_0, R_0 \circ v\} = \{R_0, V\}$
2. $R_{90} \circ \{R_0, V\} := \{R_{90} \circ R_0, R_{90} \circ V\} = \{R_{90}, D\}$
3. $R_{180} \circ \{R_0, V\} := \{R_{180} \circ R_0, R_{180} \circ V\} = \{R_{180}, H\}$
4. $R_{270} \circ \{R_0, V\} := \{R_{270} \circ R_0, R_{270} \circ V\} = \{R_{270}, F\}$

These sets do the trick:

$L_1 := \{R_0, R_{90}, R_{180}, R_{270}\}$ and $L_2 := \{H, V, D, F\}$

Christmas Eve Construction Applied to D_4

	R_0	H	R_{90}	F	R_{180}	V	R_{270}	D
R_0	R_0	H	R_{90}	F	R_{180}	V	R_{270}	D
R_{90}	R_{90}	F	R_{180}	V	R_{270}	D	R_0	H
R_{180}	R_{180}	V	R_{270}	D	R_0	H	R_{90}	F
R_{270}	R_{270}	D	R_0	H	R_{90}	F	R_{180}	V
H	H	R_0	D	R_{270}	V	R_{180}	F	R_{90}
V	V	R_{180}	F	R_{90}	H	R_0	D	R_{270}
D	D	R_{270}	V	R_{180}	F	R_{90}	H	R_0
F	F	R_{90}	H	R_0	D	R_{270}	V	R_{180}

Cayley-Sudoku Tables and Loops Constructions Rediscovered

With Kady Hossner ’11

In 2010 (and earlier), I asked, "Under what conditions on S can G be partitioned into complete sets of left coset representatives of all the required subgroups (i.e. of $g^{-1} \star S \star g$ for all $g \in G$)?"

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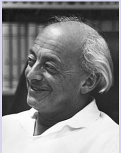
- *"You and your students have rediscovered a 1939 theorem of Reinhold Baer!"* [Emphasis added?]

Reinhold Baer 1902-1979



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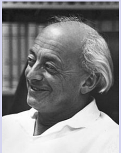

- For the record, Baer's Theorem is

  THEOREM 2.3. *The multiplication system* $(S < G; r(X)) = M$ *is a division system [i.e. quasigroup] if, and only if, the elements* $r(X)$ *form a complete set of representatives [i.e. transversal] for the right cosets of the group* G *modulo every subgroup of* G *which is conjugate to* S *in* G.

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  Roughly, it tells when coset "multiplication" gives a quasigroup–as opposed to a (factor) group.

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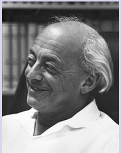

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- **Our Christmas Eve Construction is a disguised (left coset) version of Baer's theorem viewed in terms of a popular puzzle!**

Kady & Mike's Loop Examples

A loop is a set with an operation where the Cayley table is a Latin square and there is an identity (but not necessarily inverses nor associativity).

Baer also shows for any loop L, the associated left multiplication group[1] $LMult(L)$ and the subgroup fixing loop's identity $LMult(L)_e$ give a group and subgroup where Christmas Eve Construction applies. Eventually (!), this lead to examples of Cayley-Sudoku tables not known to us in 2010.
(Conjecture: all examples are of this type.)

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[1] Analogous to Cayley's Theorem, the left regular permutation representation of a group.

Another Link to the Past: Jósef Dénes 1932-2002

- THEOREM 1.5.5. *If L is the latin square representing the [Cayley] table of a group G of order n, where n is a composite number, then L can be split into a set of n $(n,1)$-complete non-trivial latin rectangles.*

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- An $(n,1)$-complete non-trivial latin rectangle is a rectangle containing each of the n elements of G exactly once. We've called them blocks. Dénes's "splitting" of G's Cayley table is a Cayley-Sudoku table!
- The theorem is true, but the proof, (in both references) is incorrect. Theorem is omitted in D & K 2nd ed. 2015. **The proof in its correct form is Keith's Construction.**

Full Disclosure: The First Cayley-Sudoku Tables (under another name and before sudoku!)

As an example see the following factorisation of the Latin square belonging to the Cayley table of a dihedral group of order 6 into complete Latin rectangles:

1	2	3	4	5	6
4	5	6	1	2	3
2	3	1	5	6	4
5	6	4	2	3	1
3	1	2	6	4	5
6	4	5	3	1	2

(Dénes, 1967, p. 262)

As an example, we show in Fig. 1.5.9 the result of carrying out the construction for the cyclic group of order 8 when regarded as the additive group of integers modulo 8, with the subgroup $\{0, 2, 4, 6\}$ as A_0. We have $A_1 = \{1, 3, 5, 7\}$ and we may take (for example) $R_1 = \{0, 1\}$, $R_2 = \{2, 5\}$, $R_3 = \{4, 3)$, $R_4 = \{6, 7\}$.

	0	2	4	6	1	3	5	7
0	0	2	4	6	1	3	5	7
1	1	3	5	7	2	4	6	0
2	2	4	6	0	3	5	7	1
5	5	7	1	3	6	0	2	4
4	4	6	0	2	5	7	1	3
3	3	5	7	1	4	6	0	2
6	6	0	2	4	7	1	3	5
7	7	1	3	5	0	2	4	6

Fig. 1.5.9

(Dénes & Keedwell, 1974, p. 49)

An Original Construction(?)

Our article “Cosets and Cayley-Sudoku Tables” contains a third construction for extending a Cayley-Sudoku table of a subgroup to a table for the big group. One referee said it was “the centerpiece of the paper." So far, we have not seen this construction elsewhere.

(see *Mathematics Magazine* Vol. 83, April 2010, pp. 130-139)

Magic Cayley-Sudoku Tables

With Rosana Mersereau ’13

Chinese Magic Square Example

This square is magic because the sum of the numbers in every row and every column and the two diagonals is 15, the magic constant for this square.

$$
\begin{array}{cc|ccc|cc}
\cline{3-5}
 & & 4 & 9 & 2 & \rightarrow & 15 \\
 & & 3 & 5 & 7 & \rightarrow & 15 \\
 & & 8 & 1 & 6 & \rightarrow & 15 \\
\cline{3-5}
 & \swarrow & \downarrow & \downarrow & \downarrow & \searrow & \\
15 & & 15 & 15 & 15 & & 15
\end{array}
$$

Allegedly, this magic square appears on the back of a sacred turtle in an ancient Chinese legend.

(BBC series "The Story of Maths" 2008, episode 2)

Magic Sudoku Table for $\mathbb{Z}_9$

1	8	0	7	5	6	4	2	3
2	3	4	8	0	1	5	6	7
6	7	5	3	4	2	0	1	8
8	4	6	5	1	3	2	7	0
7	0	2	4	6	8	1	3	5
3	5	1	0	2	7	6	8	4
5	1	3	2	7	0	8	4	6
4	6	8	1	3	5	7	0	2
0	2	7	6	8	4	3	5	1

(Lorch & Weld 2011)

In each 3×3 block, the sum–*using the group operation, addition mod 9*–of the elements in each row, each column, and each diagonal is 0.

Magic Cayley-Sudoku Tables

A **Magic Cayley-Sudoku table** is a Cayley-Sudoku table in which the blocks are magic squares, that is, the blocks are square and operating the elements in every row, column, and diagonal *using the group operation* is the same group element, called the **magic constant**.

$$\begin{array}{c|cccc|c} & & & & & \\ \hline & \star & \star & \cdots & \star & \rightarrow \\ & \star & \star & \cdots & \star & \rightarrow \\ & \vdots & \vdots & & \vdots & \vdots \\ & \star & \star & \cdots & \star & \rightarrow \\ \hline \swarrow & \downarrow & \downarrow & \cdots & \downarrow & \searrow \end{array}$$

Operating elements in the block as indicated by the arrows (directions matter) are the same.

Magic Cayley-Sudoku Table Example

	00	10	20	01	11	21	02	12	22
00	00	10	20	01	11	21	02	12	22
01	01	11	21	02	12	22	00	10	20
02	02	12	22	00	10	20	01	11	21
10	10	20	00	11	21	01	12	22	02
11	11	21	01	12	22	02	10	20	00
12	12	22	02	10	20	00	11	21	01
20	20	00	10	21	01	11	22	02	12
21	21	01	11	22	02	12	20	00	10
22	22	02	12	20	00	10	21	01	11

A Magic Cayley-Sudoku Table for $\mathbb{Z}_3 \times \mathbb{Z}_3$ with magic constant 00
($00 = (0,0)$, $10 = (1,0)$, etc.)

Counting

- How many Magic Sudoku Tables for $\mathbb{Z}_9$?

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Counting

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- 20,155,392 (Ward 2015)

Rosanna & Mike's Magic Construction

If

- G is a group of order k^2 and S is a subgroup of G of order k.
- The order of every element divides k and $S \le Z(G)$.
- (Trivial Product Property) There exists an ordering $[S] = [h_1, h_2, \ldots, h_k]$ where $h_1 \star h_2 \star \cdots \star h_k = e$.
- (Magic Shuffle Property) There exists a CSLCR $[T] = [t_1, t_2, \ldots, t_k]$ of S in G such that for every $t \in T$, $(t_1 \star t) \star (t_2 \star t) \star \cdots \star (t_k \star t) = e$.

Then

The following layout gives a Magic Cayley-Sudoku Table with magic constant e, the group identity. (In fact, pandiagonal magic.)

	$[St_1]$	$[St_2]$	$\ldots$	$[St_n]$
$[Th_1]$				
$[Th_2]$				
$\vdots$				
$[Th_k]$				

Sample Applications of the Magic Construction

1. Any abelian group of odd order k^2 where the order of every element divides k has a Magic Cayley-Sudoku Table.
2. While not all groups have Magic Cayley-Sudoku tables, every group is isomorphic to a subgroup of a group having a Magic Cayley-Sudoku table.

Other Opportunities for Undergraduate Research

1. Counting other types of Cayley-Sudoku Tables.
2. The theory of Orthogonal Cayley-Sudoku Tables and Mutually Orthogonal Sets of Cayley-Sudoku Tables is in the early stages.
3. What is the minimum number of entries needed in a Cayley-Sudoku puzzle?
4. Algorithms for producing Cayley-Sudoku puzzles?
5. For the other construction (extending a Cayley-Sudoku table of a subgroup to a table for the big group) and more open questions see "Cosets and Cayley-Sudoku Tables", *Mathematics Magazine* Vol. 83, April 2010, pp. 130-139.