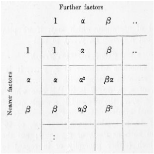
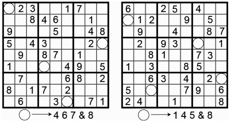
Cayley-Sudoku Tables, Loops, Quasigroups, and More Questions from Undergraduate Research

Michael Ward

XXXI Ohio State-Denison Mathematics Conference

May 2012





	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

First Cayley Table (1854) & First Sudoku Puzzle (1979) & First Cayley-Sudoku Table (2010)

Outline

1. Cayley-Sudoku Tables Review
2. Construction 2 = Baer's Theorem
3. Construction 1 = Dénes's Theorem with a Correct Proof
4. Construction 3 = ??
5. The Zassenhaus Connection
6. A Magic Cayley-Sudoku Table (time permitting)

Sudoku

Sudoku puzzles are 9×9 arrays divided into nine 3×3 sub-arrays or blocks. Digits 1 through 9 appear in some of the entries. Other entries are blank. The goal is to fill the blank entries with the digits 1 through 9 in such a way that each digit appears exactly once in each row and in each column, and in each block.

	3			4	7			
		7					6	9
2				6		4		
	6			7		5		2
4		1		8				
					3		1	
6								5
			8	2			3	
		5	9			1		7

Cayley-Sudoku Tables

A Cayley-Sudoku Table is the Cayley table of a group arranged (unconventionally) so that the body of the Cayley table has blocks containing each group element exactly once.

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

A Cayley-Sudoku table of $\mathbb{Z}_9$ (with $9 = 0$).

	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124)
(1)	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124)
(13)(24)	(13)(24)	(14)(23)	(1)	(12)(34)	(142)	(134)	(123)	(243)	(234)	(124)	(132)	(143)
(123)	(123)	(134)	(243)	(142)	(132)	(124)	(143)	(234)	(1)	(14)(23)	(12)(34)	(13)(24)
(243)	(243)	(142)	(123)	(134)	(143)	(234)	(132)	(124)	(12)(34)	(13)(24)	(1)	(14)(23)
(132)	(132)	(234)	(124)	(143)	(1)	(13)(24)	(14)(23)	(12)(34)	(123)	(142)	(134)	(243)
(143)	(143)	(124)	(234)	(132)	(12)(34)	(14)(23)	(13)(24)	(1)	(243)	(134)	(142)	(123)
(12)(34)	(12)(34)	(1)	(14)(23)	(13)(24)	(243)	(123)	(134)	(142)	(143)	(132)	(124)	(234)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	(1)	(134)	(142)	(243)	(123)	(124)	(234)	(143)	(132)
(134)	(134)	(123)	(142)	(243)	(124)	(132)	(234)	(143)	(14)(23)	(1)	(13)(24)	(12)(34)
(142)	(142)	(243)	(134)	(123)	(234)	(143)	(124)	(132)	(13)(24)	(12)(34)	(14)(23)	(1)
(234)	(234)	(132)	(143)	(124)	(13)(24)	(1)	(12)(34)	(14)(23)	(142)	(123)	(243)	(134)
(124)	(124)	(143)	(132)	(234)	(14)(23)	(12)(34)	(1)	(13)(24)	(134)	(243)	(123)	(142)

A Cayley-Sudoku table of A_4 with 6×2 blocks.

How to construct non-trivial Cayley-Sudoku tables?

("Non-trivial" meaning the blocks are not just single rows or columns.)

Construction 2

Assume H is a subgroup of G having order k and index n. Also suppose t_1H, t_2H, …, t_nH are the distinct left cosets of H in G. Arranging the Cayley table of G with columns labeled by the cosets t_1H, t_2H, …, t_nH and the rows labeled by sets $L_1, L_2, \ldots, L_k$ yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $L_1, L_2, \ldots, L_k$ are left transversals of H^g for all $g \in G$.

	t_1H	t_2H	…	t_nH
L_1				
L_2				
$\vdots$				
L_k				

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, "Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?"

Answers we knew in 2010

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, "Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?"

Answers we knew in 2010

- Not always.

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, “Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?”

Answers we knew in 2010

- Not always.
- When H is a normal subgroup, i.e. only one conjugate.

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, "Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?"

Answers we knew in 2010

- Not always.
- When H is a normal subgroup, i.e. only one conjugate.
- When H has a complement, i.e. $\exists T \leq G$ such that $G = TH$ and $T \cap H = 1$.

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, "Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?"

Answers we knew in 2010

- Not always.
- When H is a normal subgroup, i.e. only one conjugate.
- When H has a complement, i.e. $\exists T \leq G$ such that $G = TH$ and $T \cap H = 1$.
- When H has only two conjugates, i.e. $[G : N_G(H)] = 2$.

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, “Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?”

Answers we knew in 2010

- Not always.
- When H is a normal subgroup, i.e. only one conjugate.
- When H has a complement, i.e. $\exists T \leq G$ such that $G = TH$ and $T \cap H = 1$.
- When H has only two conjugates, i.e. $[G : N_G(H)] = 2$.

Answer from the 2010 audience

Interesting (?) Question in Search of an Answer

In 2010 (and earlier), I asked, "Given a subgroup H of a finite group G, under what circumstances is it possible to partition G into sets $L_1, L_2, \ldots, L_k$ where for every $g \in G$ each L_i is a left transversal of H^g?"

Answers we knew in 2010

- Not always.
- When H is a normal subgroup, i.e. only one conjugate.
- When H has a complement, i.e. $\exists T \leq G$ such that $G = TH$ and $T \cap H = 1$.
- When H has only two conjugates, i.e. $[G : N_G(H)] = 2$.

Answer from the 2010 audience

- *"You and your students have rediscovered a 1939 theorem of Reinhold Baer!"* [Emphasis added?]

Reexamine Construction 2 to See Baer's Theorem

A Cayley-Sudoku table from Construction 2 looks like

	t_1H	t_2H	$\ldots$	t_nH
L_1				
L_2				
$\vdots$				
L_k				

Look at a "row of blocks" from the table

	t_1H	t_2H	$\ldots$	t_nH
L_i				

	$t_1 H$	$t_2 H$	$\dots$	$t_n H$
L_i				

Let $L_i = \{\ell_1, \ell_2, \dots, \ell_n\}$. Expand the row labels and fill-in the rows.

	$t_1 H$	$t_2 H$	$\dots$	$t_n H$
ℓ_1	$\ell_1 t_1 H$	$\ell_1 t_2 H$		$\ell_1 t_n H$
ℓ_2	$\ell_2 t_1 H$	$\ell_2 t_2 H$	$\dots$	$\ell_2 t_n H$
$\vdots$				
ℓ_n	$\ell_n t_1 H$	$\ell_n t_2 H$	$\dots$	$\ell_n t_n H$

	t_1H	t_2H	$\dots$	t_nH
ℓ_1	$\ell_1 t_1 H$	$\ell_1 t_2 H$		$\ell_1 t_n H$
ℓ_2	$\ell_2 t_1 H$	$\ell_2 t_2 H$	$\dots$	$\ell_2 t_n H$
$\vdots$				
ℓ_n	$\ell_n t_1 H$	$\ell_n t_2 H$	$\dots$	$\ell_n t_n H$

Recall $L_i = \{\ell_1, \ell_2, \dots, \ell_n\}$ is a left transversal of H (and all its conjugates) in G, relabel the cosets.

	ℓ_1H	ℓ_2H	$\dots$	ℓ_nH
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
ℓ_2	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1 \ell_1 H$	$\ell_1 \ell_2 H$		$\ell_1 \ell_n H$
ℓ_2	$\ell_2 \ell_1 H$	$\ell_2 \ell_2 H$	$\dots$	$\ell_2 \ell_n H$
$\vdots$				
ℓ_n	$\ell_n \ell_1 H$	$\ell_n \ell_2 H$	$\dots$	$\ell_n \ell_n H$

- Each row contains the n distinct left cosets of H in G.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
ℓ_2	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

- Each row contains the n distinct left cosets of H in G.
- Proof: Just apply the left regular permutation representation of G corresponding to left multiplication by ℓ_j.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
ℓ_2	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

- Each row contains the n distinct left cosets of H in G.
- Proof: Just apply the left regular permutation representation of G corresponding to left multiplication by ℓ_j.
- Each column contains the n distinct left cosets of H in G.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
ℓ_2	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

- Each row contains the n distinct left cosets of H in G.
- Proof: Just apply the left regular permutation representation of G corresponding to left multiplication by ℓ_j.
- Each column contains the n distinct left cosets of H in G.
- Proof: The sudoku condition requires that each block contain all the elements of G.
  $\therefore$ The n cosets seen in each column must be distinct.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
ℓ_2	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

- Each row contains the n distinct left cosets of H in G.
- Proof: Just apply the left regular permutation representation of G corresponding to left multiplication by ℓ_j.
- Each column contains the n distinct left cosets of H in G.
- Proof: The sudoku condition requires that each block contain all the elements of G.
  $\therefore$ The n cosets seen in each column must be distinct.
- The body of the table is a Latin square by definition.

	$\ell_1 H$	$\ell_2 H$	$\ldots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\ldots$	$\ell_n\ell_n H$

Replace each row label ℓ_j with the coset $\ell_j H$.

	$\ell_1 H$	$\ell_2 H$	$\ldots$	$\ell_n H$
$\ell_1 H$	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
$\ell_2 H$	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\ldots$	$\ell_2\ell_n H$
$\vdots$				
$\ell_n H$	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\ldots$	$\ell_n\ell_n H$

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1 \ell_1 H$	$\ell_1 \ell_2 H$		$\ell_1 \ell_n H$
$\vdots$				
ℓ_n	$\ell_n \ell_1 H$	$\ell_n \ell_2 H$	$\dots$	$\ell_n \ell_n H$

Replace each row label ℓ_j with the coset $\ell_j H$.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
$\ell_1 H$	$\ell_1 \ell_1 H$	$\ell_1 \ell_2 H$		$\ell_1 \ell_n H$
$\ell_2 H$	$\ell_2 \ell_1 H$	$\ell_2 \ell_2 H$	$\dots$	$\ell_2 \ell_n H$
$\vdots$				
$\ell_n H$	$\ell_n \ell_1 H$	$\ell_n \ell_2 H$	$\dots$	$\ell_n \ell_n H$

- The resulting Cayley table defines a quasigroup operation on the left cosets of H in G by definition.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
ℓ_1	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
$\vdots$				
ℓ_n	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

Replace each row label ℓ_j with the coset $\ell_j H$.

	$\ell_1 H$	$\ell_2 H$	$\dots$	$\ell_n H$
$\ell_1 H$	$\ell_1\ell_1 H$	$\ell_1\ell_2 H$		$\ell_1\ell_n H$
$\ell_2 H$	$\ell_2\ell_1 H$	$\ell_2\ell_2 H$	$\dots$	$\ell_2\ell_n H$
$\vdots$				
$\ell_n H$	$\ell_n\ell_1 H$	$\ell_n\ell_2 H$	$\dots$	$\ell_n\ell_n H$

- The resulting Cayley table defines a quasigroup operation on the left cosets of H in G by definition.
- Baer names this system $(H < G; r(X))$ with function $r(X)$ referring to the choice of transversals. Here $r(\ell_i H) = \ell_i$.

Recapitulation

- From a Cayley-Sudoku table from Construction 2

Recapitulation

- From a Cayley-Sudoku table from Construction 2
- each “row of blocks” from the table

	$t_1 H$	$t_2 H$	$\dots$	$t_n H$
L_i				

Recapitulation

- From a Cayley-Sudoku table from Construction 2
- each "row of blocks" from the table

	t_1H	t_2H	$\dots$	t_nH
L_i				

- leads to a quasigroup $(H < G; r(X))$ on the left cosets of H in G.

Recapitulation

- From a Cayley-Sudoku table from Construction 2
- each "row of blocks" from the table

	$t_1 H$	$t_2 H$	$\dots$	$t_n H$
L_i				

- leads to a quasigroup $(H < G; r(X))$ on the left cosets of H in G.
- The converse is also true. So ...

Recapitulation

- From a Cayley-Sudoku table from Construction 2
- each "row of blocks" from the table

	t_1H	t_2H	$\ldots$	t_nH
L_i				

- leads to a quasigroup $(H < G; r(X))$ on the left cosets of H in G.
- The converse is also true. So …

	t_1H	t_2H	$\ldots$	t_nH
L_1				
$\vdots$				
L_k				

- **gives a Cayley-Sudoku table.**

$\Longleftrightarrow$ **Each L_j is a left transversal of H^g for all $g \in G$.**

$\Longleftrightarrow$ **Each row of blocks leads to a quasigroup $(H < G; r(X))$ on the left cosets of H in G.**

Baer's Theorem

The last equivalence on the previous slide is Baer's Theorem.

Baer's Theorem

The last equivalence on the previous slide is Baer's Theorem.

- For the record,

  THEOREM 2.3. *The multiplication system* $(S < G; r(X)) = M$ *is a division system [i.e. quasigroup] if, and only if, the elements* $r(X)$ *form a complete set of representatives [i.e. transversal] for the right cosets of the group* G *modulo every subgroup of* G *which is conjugate to* S *in* G.

  –R. Baer, Nets and Groups, *Transactions of the AMS*, 1939.

Baer's Theorem

The last equivalence on the previous slide is Baer's Theorem.

- For the record,

  THEOREM 2.3. *The multiplication system* $(S < G; r(X)) = M$ *is a division system [i.e. quasigroup] if, and only if, the elements* $r(X)$ *form a complete set of representatives [i.e. transversal] for the right cosets of the group* G *modulo every subgroup of* G *which is conjugate to* S *in* G.

  –R. Baer, Nets and Groups, *Transactions of the AMS*, 1939.

- **Our Construction 2 is, therefore, just a (left-handed) version of Baer's theorem viewed in terms of a popular puzzle!**

Remarks on Baer & Construction 2

1. One of the sets L_j contains 1. The corresponding quasigroup will have an identity $1H$. That one is a loop (i.e. quasigroup with identity).

[1] Analogous to the left regular permutation representation of a group.

Remarks on Baer & Construction 2

1. One of the sets L_j contains 1. The corresponding quasigroup will have an identity $1H$. That one is a loop (i.e. quasigroup with identity).
2. Baer also shows for any loop L, the left multiplication group[1] $LMult(L)$ and the stabilizer of the loop's identity $LMult(L)_e$ give a group and subgroup where Construction 2 applies. Eventually (!), this lead to examples of Cayley-Sudoku tables not known to us in 2010 (with Kady Hossner WOU '11).

[1] Analogous to the left regular permutation representation of a group.

Remarks on Baer & Construction 2

1. One of the sets L_j contains 1. The corresponding quasigroup will have an identity $1H$. That one is a loop (i.e. quasigroup with identity).
2. Baer also shows for any loop L, the left multiplication group[1] $LMult(L)$ and the stabilizer of the loop's identity $LMult(L)_e$ give a group and subgroup where Construction 2 applies. Eventually (!), this lead to examples of Cayley-Sudoku tables not known to us in 2010 (with Kady Hossner WOU '11).
3. Baer further shows how to think of these ideas geometrically in terms of nets or 3-webs. No time for that today.

[1] Analogous to the left regular permutation representation of a group.

Construction 1 or Keith's Construction

Let G be a finite group. Assume H is a subgroup of G having order k and index n. If $Hg_1, Hg_2, \ldots, Hg_n$ are the n distinct right cosets of H in G, then arranging the Cayley table of G with columns labeled by the cosets $Hg_1, Hg_2, \ldots, Hg_n$ and the rows labeled by sets $T_1, T_2, \ldots, T_k$ (as in the table) yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $T_1, T_2, \ldots, T_k$ partition G into left transversals of H in G.

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

Is this also a rediscovery of an older result?

Dénes's Theorem

- THEOREM 1.5.5. *If L is the latin square representing the multiplication table of a group G of order* n*, where* n *is a composite number, then L can be split into a set of* n $(n, 1)$*-complete non-trivial latin rectangles.*

  –J. Dénes and A. D. Keedwell, *Latin Squares and Their Applications*, 1974.

  –J. Dénes, Algebraic and Combinatorial Characterization of Latin Squares I, *Mathematica Slovaca*, 1967.

Dénes's Theorem

- THEOREM 1.5.5. *If L is the latin square representing the multiplication table of a group G of order n, where n is a composite number, then L can be split into a set of n* $(n, 1)$*-complete non-trivial latin rectangles.*

  –J. Dénes and A. D. Keedwell, *Latin Squares and Their Applications*, 1974.

  –J. Dénes, Algebraic and Combinatorial Characterization of Latin Squares I, *Mathematica Slovaca*, 1967.

- An $(n, 1)$-complete non-trivial latin rectangle is a rectangle containing each of the n elements of G exactly once. We've called them blocks. Dénes's "splitting" of G's Cayley table is a Cayley-Sudoku table!

Dénes's Theorem

- THEOREM 1.5.5. *If L is the latin square representing the multiplication table of a group G of order n, where n is a composite number, then L can be split into a set of n $(n,1)$-complete non-trivial latin rectangles.*

  –J. Dénes and A. D. Keedwell, *Latin Squares and Their Applications*, 1974.

  –J. Dénes, Algebraic and Combinatorial Characterization of Latin Squares I, *Mathematica Slovaca*, 1967.

- An $(n,1)$-complete non-trivial latin rectangle is a rectangle containing each of the n elements of G exactly once. We've called them blocks. Dénes's "splitting" of G's Cayley table is a Cayley-Sudoku table!
- The theorem is true, but the proof (in both references) is incorrect.

Dénes's Error

Dénes's Error

- Incorrect proof.
  Take a proper non-trivial subgroup H of G and arrange the Cayley table in this way

	Hg_1	Hg_2	$\dots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

  where $T_1, T_2, \dots, T_k$ partition G into **right** transversals of H in G.

Dénes's Error

- Incorrect proof.
  Take a proper non-trivial subgroup H of G and arrange the Cayley table in this way

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

  where $T_1, T_2, \ldots, T_k$ partition G into **right** transversals of H in G.

- Examples show the resulting blocks might not contain each element of G exactly once. **Left** transversals are needed.

Dénes's Error

- Incorrect proof.
  Take a proper non-trivial subgroup H of G and arrange the Cayley table in this way

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

  where $T_1, T_2, \ldots, T_k$ partition G into **right** transversals of H in G.
- Examples show the resulting blocks might not contain each element of G exactly once. **Left** transversals are needed.
- **Our Construction 1 is Dénes's theorem with a correct proof!**

Construction 3: Extending Cayley-Sudoku Tables

Let G be a finite group with a subgroup A. Let $C_1, C_2, \ldots, C_k$ partition A and $R_1, R_2, \ldots R_n$ partition A such that the following table is a Cayley-Sudoku table of A.

	C_1	C_2	$\ldots$	C_k
R_1				
R_2				
$\vdots$				
R_n				

Construction 3, continued

If $\{l_1, l_2, \ldots, l_t\}$ and $\{r_1, r_2, \ldots r_t\}$ are left and right transversals, respectively, of A in G, then arranging the Cayley table of G with columns labeled with the sets $C_i r_j$, $i = 1, \ldots, k$, $j = 1, \ldots, t$ and the b^{th} block of rows labeled with $l_j R_b$, $j = 1, \ldots, t$, for $b = 1, \ldots, n$ yields a Cayley-Sudoku table of G with blocks of dimension $tk \times n$.

	$C_1 r_1$	$C_2 r_1$	$\ldots$	$C_k r_1$	$C_1 r_2$	$\ldots$	$C_k r_2$	$\ldots$	$C_1 r_t$	$\ldots$	$C_k r_t$
$l_1 R_1$ $l_2 R_1$ $\vdots$ $l_t R_1$											
$l_1 R_2$ $\vdots$ $l_t R_2$											
$\vdots$											
$l_1 R_n$ $\vdots$ $l_t R_n$											

Is Construction 3 also a rediscovery of an older result?

The Zassenhaus Connection

From “Historical notes on loop theory” by H. O. Pflugfelder,

“On the algebraic scene, brilliant algebraists happened to be in Hamburg at the time, such as Erich Hecke, a student of Hilbert; Emil Artin; and Artin’s students, Max Zorn and *Hans Zassenhaus* . . . Bol gives an example by *Zassenhaus*. This example (of order 81) was the first example of a non-associative commutative Moufang loop . . . It was *Zassenhaus*, again, who soon constructed the first example of a right Bol loop.”

–*Commentationes Mathematicae Universitatis Carolinae*, 2000, emphasis added

(Loops played a central role in the Honors Thesis of Kady Hossner WOU ’11 on Construction 2, but not as much of a role in this talk as expected.)

Thank you!!

To read about the constructions, more open questions for undergraduate exploration, and work a Cayley-Sudoku puzzle see Carmichael, Schloeman, and Ward, Cosets and Cayley-Sudoku Tables, *Mathematics Magazine* 83 (April 2010), pp. 130-139.

Bonus Slide: A Magic Cayley-Sudoku Table

In this Cayley-Sudoku table of $\mathbb{Z}_3 \times \mathbb{Z}_3$ with (a, b) abbreviated ab

	00	10	20	01	11	21	02	12	22
00	00	10	20	01	11	21	02	12	22
01	01	11	21	02	12	22	00	10	20
02	02	12	22	00	10	20	01	11	21
10	10	20	00	11	21	01	12	22	02
11	11	21	01	12	22	02	10	20	00
12	12	22	02	10	20	00	11	21	01
20	20	00	10	21	01	11	22	02	12
21	21	01	11	22	02	12	20	00	10
22	22	02	12	20	00	10	21	01	11

the sum of each row, each column, and each diagonal in each block is 00. **Magic!**