On a Conjecture of Alvis

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Abstract. We exhibit for each integer $n \geq 15$ an ordinary irreducible character of the symmetric group S_n, which restricts irreducibly to A_n, with the property that its degree is divisible by every prime less than or equal to n, thereby proving a conjecture of D. L. Alvis.

1 Introduction

In [1], D. L. Alvis made the following conjecture:

> Every alternating group A_n, with $n \geq 15$, has an ordinary irreducible character whose degree is divisible by every prime less than or equal to n.

We prove this conjecture by producing for each integer $n \geq 15$—in a systematic way for $n \geq 49$—an ordinary irreducible character χ_n of the symmetric group S_n, which restricts irreducibly to A_n, with this property.

This implies that the diameter of the degree graph of A_n is 1 for $n \geq 15$, improving the result of Lewis and White [4, Lemma 2.4].

The characters of S_n are parametrized by the partitions of n [3, Theorem 2.1.11]. Denote the character of S_n corresponding to the partition α of n by χ_α. Each partition α of n has an associated diagram $[\alpha]$ [2, Definition 3.1] and an associated hook graph [2, Definition 18.2]. For example, if $\alpha = (5, 3, 3)$ then

$$[\alpha] = \begin{array}{ccccc} x & x & x & x & x \\ x & x & x & & \\ x & x & x & & \end{array} \qquad \text{and its hook graph is} \qquad \begin{array}{ccccc} 7 & 6 & 5 & 2 & 1 \\ 4 & 3 & 2 & & \\ 3 & 2 & 1 & & \end{array}.$$

If $[\alpha]$ is a diagram, the *conjugate* diagram $[\alpha']$ is obtained by interchanging the rows and columns in $[\alpha]$. Then α' is the partition conjugate to α [2, Definition 3.5]. If $\alpha' \neq \alpha$, then χ_α restricts irreducibly to A_n [3, Theorem 2.5.7]. Finally, if h is the product of all the hook lengths in the hook graph of the diagram $[\alpha]$, then $\chi_\alpha(1) = n!/h$ [2, Theorem 20.1].

For each integer $n \geq 49$, we now define a partition α_n of n such that $\chi_{\alpha_n}(1)$ is divisible by every prime $p \leq n$.

Definition 1. For $n \geq 49$, define k to be the unique positive integer satisfying $k^2 \leq n < (k+1)^2$, that is, $k = \lfloor \sqrt{n} \rfloor$. (Strictly speaking we should use a notation such as k_n to indicate dependence on n, but we will not do this.)

Case 1: k is odd.

For $j = 1, \ldots, k$, define $a_j = k + (k-1)/2 + 1 - j$. Define the partition α_n of n to be

$(a_1, a_2, \ldots, a_k)$ if $n = k^2$;

$(a_1+1, a_2+1, \ldots, a_j+1, a_{j+1}, \ldots, a_k)$ if $n = k^2 + j$ with $1 \le j \le k$;
$(a_1+2, a_2+2, \ldots, a_j+2, a_{j+1}+1, \ldots, a_k+1)$ if $n = k^2+k+j$ with $1 \le j \le k$.
Case 2: k is even.
Define

$$a_i = \begin{cases} k + k/2 - i, & \text{if } 1 \le i \le k/2; \\ k + k/2 + 1 - i, & \text{if } k/2 + 1 \le i \le k. \end{cases}$$

Define the partition α_n of n to be
$(a_1, a_2, \ldots, a_k)$ if $n = k^2$;
$(a_1, \ldots, a_{k/2-j}, a_{k/2-j+1}+1, \ldots, a_{k/2}+1, a_{k/2+1}, \ldots, a_k)$ if $n = k^2 + j$ with $1 \le j \le k/2 - 1$;
$(a_1, a_2+1, \ldots, a_{k/2}+1, a_{k/2+1}, \ldots, a_{k-1}, a_k+1)$ if $n = k^2 + k/2$;
$(a_1, a_2+1, \ldots, a_{k/2}+1, a_{k/2+1}, \ldots, a_{k-j-1}, a_{k-j}+1, \ldots, a_k+1)$ if $n = k^2 + k/2 + j$ with $1 \le j \le k/2 - 1$;
$(a_1+1, a_2+1, \ldots, a_k+1)$ if $n = k^2 + k$
$(a_1+1, \ldots, a_{k/2-j}+1, a_{k/2-j+1}+2, \ldots, a_{k/2}+2, a_{k/2+1}+1, \ldots, a_k+1)$ if $n = k^2 + k + j$ with $1 \le j \le k/2 - 1$;
$(a_1+1, a_2+2, \ldots, a_{k/2}+2, a_{k/2+1}+1, \ldots, a_{k-1}+1, a_k+2)$ if $n = k^2+k+k/2$;
$(a_1+1, a_2+2, \ldots, a_{k/2}+2, a_{k/2+1}+1, \ldots, a_{k-j-1}+1, a_{k-j}+2, \ldots, a_k+2)$ if $n = k^2 + k + k/2 + j$ with $1 \le j \le k/2 - 1$;
$(a_1+2, \ldots, a_k+2)$ if $n = k^2 + 2k$.

Notice that in both cases the partition α_{k^2+k+j} is obtained by adding 1 to each number in α_{k^2+j}, $0 \le j < k$, and α_{k^2+2k} is obtained by adding 2 to each number in α_{k^2}.
The following example for $k = 7$ illustrates the scheme.

49: (10, 9, 8, 7, 6, 5, 4)
50: (11, 9, 8, 7, 6, 5, 4)
51: (11, 10, 8, 7, 6, 5, 4)
52: (11, 10, 9, 7, 6, 5, 4)
53: (11, 10, 9, 8, 6, 5, 4)
54: (11, 10, 9, 8, 7, 5, 4)
55: (11, 10, 9, 8, 7, 6, 4)
56: (11, 10, 9, 8, 7, 6, 5)
57: (12, 10, 9, 8, 7, 6, 5)
58: (12, 11, 9, 8, 7, 6, 5)
59: (12, 11, 10, 8, 7, 6, 5)
60: (12, 11, 10, 9, 7, 6, 5)
61: (12, 11, 10, 9, 8, 6, 5)
62: (12, 11, 10, 9, 8, 7, 5)
63: (12, 11, 10, 9, 8, 7, 6)

The following example for $k = 8$ illustrates the above scheme.

64: (11,10,9,8,8,7,6,5)
72: (12,11,10,9,9,8,7,6)

65: (11,10,9,9,8,7,6,5)	73: (12,11,10,10,9,8,7,6)
66: (11,10,10,9,8,7,6,5)	74: (12,11,11,10,9,8,7,6)
67: (11,11,10,9,8,7,6,5)	75: (12,12,11,10,9,8,7,6)
68: (11,11,10,9,8,7,6,6)	76: (12,12,11,10,9,8,7,7)
69: (11,11,10,9,8,7,7,6)	77: (12,12,11,10,9,8,8,7)
70: (11,11,10,9,8,8,7,6)	78: (12,12,11,10,9,9,8,7)
71: (11,11,10,9,9,8,7,6)	79: (12,12,11,10,10,9,8,7)
	80: (13,12,11,10,10,9,8,7)

For $n \geq 49$, we will write χ_n for χ_{α_n}. Since the number of rows k of $[\alpha_n]$ is strictly less than the number of columns, $\alpha_n \neq \alpha_n'$ and therefore χ_n restricts irreducibly to A_n.

Denote the largest hook length of the hook graph of the diagram $[\alpha_n]$ by γ_n. In our proof we will partition the set of primes less than or equal to n into 3 parts:

1. primes in the range $[2, k]$, which we will call small;
2. primes in the range $(k, \gamma_n]$, which we will call middle;
3. primes in the range $(\gamma_n, n]$, which we will call large.

If a prime p is large, it occurs in $n!$ but not in h and thus divides $n!/h$. We will handle small primes fairly easily by adapting an argument of [1]. Most of our energy will be spent on the middle primes. Since we will show in the coming sections that $\gamma_n < 3k$, the only multiples of a middle prime p which can occur as hook lengths are p and $2p$.

For the middle primes, we consider two cases depending upon the parity of k and subcases depending on the residue class of k mod 4. The strategy of the proofs is the same in all the cases, so we outline it here. It is clear from the definition that the hook lengths in any row or any column of a hook graph decrease as we move to the right along the row or down the column. If the lowest row on which a hook length ℓ appears is row m, then the number of hook lengths equal to ℓ is at most m because a hook length can appear at most once in a row. In particular, when ℓ is p and when ℓ is $2p$ where p is a middle prime, then the corresponding values of m allow us to bound the number of factors of p in h. However, in some cases that bound is not sufficient to prove p divides $n!/h$. Therefore, in those cases, we sharpen the bound. We shall see in the hook graphs that a hook length $\ell > k$, where k is odd, always occurs in one of the first three columns of the lowest row m in which it occurs. If we can produce a row u in which ℓ does not appear and for which the values in the first three columns are all greater than ℓ, then $u < m$, and so the number of hook lengths equal to ℓ is strictly less than m. When k is even, the technique is the same, but it will suffice to look at the first two columns. The sharper bounds thus obtained are sufficient to prove that middle primes divide $n!/h = \chi_n(1)$.

2 Middle Primes

For the rest of the paper, we fix the meanings of n, k, α_n, γ_n and χ_n established in the introduction. When referring to diagrams of partitions and their corresponding hook graphs, we follow the convention of numbering rows from top to bottom, columns from left to right.

It is clear from considering the leftmost three columns of the hookgraph for α_n—the leftmost two columns suffice when $n = k^2$, $k^2 + k$ or $k^2 + 2k$ or when k is even—and the bottom row that every integer in $[1, \gamma_n]$ occurs as a hooklength (see the proofs of Lemmas 4 and 7 below where the relevant columns are listed). The following notation will be useful.

Definition 2. For positive integers n and p, $LR(n, p)$ equals the largest integer r for which p is in row r. In other words, $LR(n, p)$ equals the number of the lowest row of the hook graph of α_n in which p appears. When p does not appear in the hook graph for α_n, $LR(n, p)$ is defined to be 0. We define $NH(n, p)$ to be the number of hook lengths in the hook graph for α_n equal to p.

Lemma 1. *If p is an integer, $k \geq 7$, and $k^2 \leq n \leq k^2 + 2k - 1$, then $LR(n + 1, p) \geq LR(n, p)$.*

Proof. The diagram for α_{n+1} is obtained from the diagram for α_n by adding a single node at the right of one row, call it row v, in the diagram for α_n. Hook lengths on rows $v + 1$ to k are unaffected by this. Hook lengths on row v are all increased by 1 with an extra hook length 1 added at the right. Hook lengths in the column above this new 1 are all increased by 1. Since $k \geq 7$, $a_k > 3$ and so hook lengths in columns 1 to 3 of rows 1 to $v - 1$ are unaffected. Furthermore, the lowest appearance of a hooklength occurs in columns 1 to 3 when k is odd, columns 1 to 2 when k is even (the proofs of Lemmas 4 and 7 list the relevant columns). The upshot of all of this is that exactly one hook length, the leftmost one in row v of the hook graph for α_{n+1} has a new lowest appearance, whereas as the lowest appearance of every other hook length is the same for both the hook graphs for α_n and α_{n+1}. $\square$

Two routine calculations will be used repeatedly. We record them as a lemmas for easy reference.

Lemma 2. *Suppose i, j, k, l are integers with $i \geq 1$ and $k \geq 2l - 3j - 12$, then $(k + 2i + j)(k - 3i + l) \leq k^2 + (j + l - 1)k + (l - 3)(j + 2)$.*

Proof. Simply expand and complete the square on i.

$$\begin{aligned}(k+2i+j)(k-3i+l) &= k^2 + (j+l)k + jl - 6i^2 - (k - 2l + 3j)i \\ &= k^2 + (j+l)k + jl + \frac{(k-2l+3j)^2}{24} - 6\left(i + \frac{(k-2l+3j)}{12}\right)^2 \\ &\leq k^2 + (j+l)k + jl + \frac{(k-2l+3j)^2}{24} - 6\left(1 + \frac{(k-2l+3j)}{12}\right)^2 \\ &= k^2 + (j+l-1)k + (l-3)(j+2).\end{aligned}$$

$\square$

Lemma 3. *Suppose i, j, k, l are integers, then*

$$(k+2i+j)\left(\frac{3k+l}{4} - i + 1\right) \leq \frac{25k^2 + (20j + 10l + 40)k + (2j + l + 4)^2}{32}.$$

Proof. Complete the square again.

$$\begin{aligned}(k+2i+j)\left(\frac{3k+l}{4} - i + 1\right) &= \frac{25k^2 + (20j + 10l + 40)k + (2j + l + 4)^2}{32} \\ &\quad - 2\left(i - \frac{k - 2j + l + 4}{8}\right)^2 \\ &\leq \frac{25k^2 + (20j + 10l + 40)k + (2j + l + 4)^2}{32}.\end{aligned}$$

$\square$

2.1 Middle Primes when k is Odd

Lemma 4. *Suppose k is an odd integer and $k \geq 7$. If $p = k + 2i$ is an odd integer in $(k, \gamma_n]$, then we have the following values of γ_n and $LR(n,p)$, depending upon the residue class of $k \mod 4$.*

$k \equiv 1 \mod 4$			
n	γ_n	*Parity of* γ_n	$LR(n,p)$
k^2	$\frac{5k-3}{2}$	*odd*	$\dfrac{3k+1}{4} - i$
$k^2 + j,\ 1 \leq j \leq k$	$\frac{5k-1}{2}$	*even*	$\dfrac{3k+1}{4} - i$
$k^2 + k + j,\ 1 \leq j \leq k$	$\frac{5k+1}{2}$	*odd*	$\begin{cases} \frac{3k+1}{4} - i, & \textit{if } p < \frac{5k+5}{2} - 2j \\ \frac{3k+5}{4} - i, & \textit{if } p \geq \frac{5k+5}{2} - 2j \end{cases}$

$k \equiv 3 \mod 4$			
n	γ_n	*Parity of* γ_n	$LR(n,p)$
k^2	$\frac{5k-3}{2}$	*even*	$\dfrac{3k-1}{4} - i$
$k^2 + j,\ 1 \leq j < k$	$\frac{5k-1}{2}$	*odd*	$\begin{cases} \frac{3k-1}{4} - i, & \textit{if } p < \frac{5k+3}{2} - 2j \\ \frac{3k+3}{4} - i & \textit{if } p \geq \frac{5k+5}{2} - 2j \end{cases}$
$k^2 + k + j,\ 0 \leq j \leq k$	$\frac{5k+1}{2}$	*even*	$\dfrac{3k+3}{4} - i$

Proof. From the definition of α_n, γ_n equals the first number of α_n plus $k-1$. Thus, the values and parities of γ_n are easy to calculate.

We begin with the case $n = k^2 + j$, $1 \leq j < k$. By direct computation, the first three columns of the hook graph for α_n are:

$$\begin{array}{ccc}
\gamma_n & \gamma_n - 1 & \gamma_n - 2 \\
\gamma_n - 2 & \gamma_n - 3 & \gamma_n - 4 \\
\cdots & \cdots & \cdots \\
\gamma_n - 2(j-1) & \gamma_n - (2j-1) & \gamma_n - 2j \\
\hline
\gamma_n - 2j - 1 & \gamma_n - 2j - 2 & \gamma_n - 2j - 3 \\
\cdots & \cdots & \cdots \\
\gamma_n - 2k + 3 & \gamma_n - 2k + 2 & \gamma_n - 2k + 1 \\
\gamma_n - 2k + 1 & \gamma_n - 2k & \gamma_n - 2k - 1.
\end{array}$$

For reference, row j is underlined. Observe that the hook lengths in rows 1 through j of columns 1 and 3 along with those of rows $j+1$ through k have the same parity as γ_n, while all the other hook lengths in those columns have the opposite parity.

Furthermore, those columns contain the integers from γ_n to $\gamma_n - 2k - 1$ and $\gamma_n - 2k - 1 \leq k$. Thus, the lowest appearance of p in the hook graph of α_n is in one of those first three columns.

Suppose $k \equiv 1 \mod 4$. The odd integers $\gamma_n - 1$ to $\gamma_n - (2j-1)$ occur in rows 1 to j of column 2. The odd integers $\gamma_n - (2j+1)$ to $\gamma_n - (2k-1)$ occur in rows $j+1$ to k of column 1. Thus, the lowest appearance of the odd integer $\gamma_n - (2m-1)$ for $1 \leq m \leq k$ is in row m. Since p is such an odd integer, the lowest appearance of $p = k + 2i$ is in row m where $k+2i = \gamma_n - (2m-1) = \frac{5k-1}{2} - (2m-1)$. Therefore, $m = \frac{3k+1}{4} - i = LR(n,p)$, as was to be shown.

Next suppose $k \equiv 3 \mod 4$. This time the odd integers γ_n to $\gamma_n - 2(j-1)$ occur in rows 1 to j of column 1, the odd integer $\gamma_n - 2j$ occurs in row j of column 3, and the odd integers $\gamma_n - 2(j+1)$ to $\gamma_n - 2k$ occur in rows $j+1$ to k of column 2. Thus, the lowest appearance of the odd integer $\gamma_n - 2m$ is in row $m+1$ when $0 \leq m \leq j-1$, whereas the lowest appearance is in row m when $j \leq m \leq k$. Now p is such an odd integer and $p = \gamma_n - 2m = \frac{5k-1}{2} - 2m$ yields $m = \frac{3k-1}{4} - i$. Therefore,

$$LR(n,p) = \begin{cases} m = \frac{3k-1}{4} - i, & \text{if } p < \gamma_n - 2(j-1) = \frac{5k+3}{2} - 2j \\ m+1 = \frac{3k+3}{4} - i, & \text{if } p \geq \gamma_n - 2(j-1) = \frac{5k+3}{2} - 2j. \end{cases}$$

We now turn to the case $n_1 = k^2 + k + j$, $1 \leq j < k$, retaining the above notation $n = k^2 + j$. Although the case for n_1 may be handled by similar methods, we will show how it may be obtained recursively from the case for n.

From the definition of our partitions, we see that the diagram $[\alpha_{n_1}]$ is obtained from $[\alpha_n]$ simply by adding one node per row. Therefore, $\gamma_{n_1} = \gamma_n + 1$ and the hook graph for $[\alpha_{n_1}]$ is obtained from the hook graph for α_n by adjoining a column on the left. More specifically, with row j underlined for reference, the first four columns of the hook graph for α_{n_1} are:

$$\begin{array}{cccc}
\gamma_n+1 & \gamma_n & \gamma_n-1 & \gamma_n-2 \\
\gamma_n-1 & \gamma_n-2 & \gamma_n-3 & \gamma_n-4 \\
\dots & \dots & \dots & \dots \\
\gamma_n-2j+3 & \gamma_n-2j+2 & \gamma_n-2j+1 & \gamma_n-2j \\
\hline
\gamma_n-2j & \gamma_n-2j-1 & \gamma_n-2j-2 & \gamma_n-2j-3 \\
\dots & \dots & \dots & \dots \\
\gamma_n-2k+4 & \gamma_n-2k+3 & \gamma_n-2k+2 & \gamma_n-2k+1 \\
\gamma_n-2k+2 & \gamma_n-2k+1 & \gamma_n-2k & \gamma_n-2k-1.
\end{array}$$

Let p be an odd integer in $(k, \gamma_{n_1}]$. By comparing the hook lengths in the first column with their positions in columns 2 - 4, in other words, by comparing them with their positions in the hook graph for $[\alpha_n]$, we see the following. When $k \equiv 1 \mod 4$,

$$LR(n_1,p) = \begin{cases} LR(n,p) & \text{if } p < \gamma_n - 2j + 3 = \frac{5k+5}{2} - 2j \\ LR(n,p)+1 & \text{if } p \geq \gamma_n - 2j + 3 = \frac{5k+5}{2} - 2j. \end{cases}$$

Note that this formula is valid for the "new" odd integer $\gamma_n + 1$ because $LR(n, \gamma_n + 1)$ was defined to be zero. When $k \equiv 3 \mod 4$,

$$LR(n_1,p) = \begin{cases} LR(n,p)+1 & \text{if } p < \gamma_n - 2j + 2 = \frac{5k+3}{2} - 2j \\ LR(n,p) & \text{if } p \geq \gamma_n - 2j + 2 = \frac{5k+3}{2} - 2j. \end{cases}$$

Substituting the values for $LR(n,p)$ from the previous case completes the proof here.

The remaining cases are for k^2, k^2+k, and k^2+2k. The proof for k^2 is similar to k^2+j (and easier). Then one may do k^2+k and k^2+2k directly or by using the recursive method as above.

□

Lemma 5. *Suppose k is an odd integer and $k \geq 7$. If $p = k + 2i$ is an odd integer in $(k, \gamma_n]$, then we have the following upper bounds on $NH(n,p)$, depending upon the residue class of $k \mod 4$.*

$k \equiv 1 \mod 4$		
n	*Bound*	*Strict when*
k^2+j, $0 \leq j \leq k-1$	$\frac{3k+1}{4} - i$	$0 < j$ *and* $p \in (k, 2k-1]$
k^2+k	$\frac{3k+1}{4} - i$	–
k^2+k+j, $1 \leq j \leq k$	$\frac{3k+5}{4} - i$	$j < k$ *and* $p \in (k, 2k-1]$

$k \equiv 3 \mod 4$		
n	*Bound*	*Strict when*
$k^2+j,\ 0 \leq j \leq k-1$	$\frac{3k+3}{4} - i$	$p \in (k, 2k-1]$
k^2+k	$\frac{3k+3}{4} - i$	–
$k^2+k+j,\ 1 \leq j \leq k$	$\frac{3k+3}{4} - i$	$j < k$ *and* $p \in (k, 2k-1]$

Proof. Since each hook length occurs at most per row, it is clear that $NH(n,p) \leq LR(n,p)$. Thus, the bounds follow immediately from Lemma 4 (using the largest value of $LR(n,p)$ in cases where it depends on p).

As outlined in the introduction, we show the bound is strict in certain cases by finding r such that $r < LR(n,p)$ and row r of the hook graph of α_n does not contain p.

Suppose $n = k^2 + j$, $0 \leq j \leq k-1$.

Referring to Lemma 4 again, we see the bound is strict when $k \equiv 3 \mod 4$ and $j = 0$.

Assume $0 < j$. The hook graph for α_n contains a $j \times (k-j+1)$ block of even integers

$$\begin{matrix} 2k & 2(k-1) & \dots & 2j \\ 2(k-1) & 2(k-2) & \dots & 2(j-1) \\ \vdots & \vdots & \ddots & \vdots \\ 2(k-j+1) & 2(k-j) & \dots & 2. \end{matrix}$$

beginning with the occurrence of $2k$ in row 1 of the hook diagram.

Further assume $p \in (k, 2k-1]$. Then $p-1$ and $p+1$ are among the consecutive even integers from $2k$ to 2 which appear across the first row and down the last column of the block. Thus, at least one row of the block, say row r, contains both $p-1$ and $p+1$. Since hook lengths in any row are strictly decreasing, it follows that row r of the hook graph of α_n does not contain p and that the values in the first three columns of row r are all greater that p. In particular, $r < LR(n,p)$. Thus, the bound is strict in this case.

For $n_1 = k^2+k+j$, $1 \leq j \leq k-1$, and $p \in (k, 2k-1]$, compare the hook graph for α_{n_1} to that of α_n as in the proof of Lemma 4. We see that row r of the hook graph for α_{n_1} still does not contain p and that $r < LR(n,p) \leq LR(n_1,p)$. Once again, the bound is strict.

□

Lemma 6. *Suppose k is an odd integer and $k \geq 7$. If $p = k + 2i$ and $2p$ are integers in $(k, \gamma_n]$, then we have the following bounds on $NH(n, 2p)$.*

	$k \equiv 1 \mod 4$		$k \equiv 3 \mod 4$
n	*Bound*	*Strict when*	*Bound*
$k^2+j,\ 0 \le j \le k$	$\frac{k+3}{4} - 2i$	$j=0$	$\frac{k+1}{4} - 2i$
$k^2+k+j,\ 1 \le j \le k$	$\frac{k+3}{4} - 2i$	–	$\frac{k+5}{4} - 2i$

Proof. By repeated application of Lemma 1, we see that the $LR(k^2+2k, 2p) \ge LR(k^2+j, 2p)$ with $0 \le j < 2k$ and $LR(k^2+k, 2p) \ge LR(k^2+j, 2p)$ with $0 \le j < k$.

The first two columns of the hook graph for $n = k^2$, $n = k^2+k$, and $n = k^2+2k$ are

$$\begin{array}{cc} \gamma_n & \gamma_n - 1 \\ \gamma_n - 2 & \gamma_n - 3 \\ \cdots & \cdots \\ \gamma_n - 2(k-1) & \gamma_n - 2(k-1) - 1. \end{array}$$

.

Assume $k \equiv 1 \mod 4$ and $n = k^2+2k$. The first column consists entirely of odd integers while the second consists entirely of even integers. Thus the lowest occurrence of $\gamma_n - 2(m-1)$ with $1 \le m \le k$ is on row m. Now $2(k+2i) \ge 2k+4 > (k+3)/2 = \gamma_n - 2(k-1) - 1$. Thus the lowest occurrence of $2p$ is on row m where

$$2(k+2i) = \frac{5k+3}{2} - 2m,$$

which yields $LR(k^2+2k, 2p) = m = (k+3)/4 - 2i$.

When $n = k^2$, one can check that the lowest occurrence of $2p$ is on row $(k-1)/4 - 2i$, so our bound is strict in this case as claimed.

For $k \equiv 3 \mod 4$, follow the same procedure using the hook graph for α_{k^2+k} when $n = k^2+j$, $0 \le j \le k$, and the hook graph for α_{k^2+2k} when $n = k^2+k+j$, $1 \le j \le k$. □

Proposition 1. *Suppose that $k^2 \le n < (k+1)^2$ with k an odd integer, $k \ge 7$, and p is a prime in $(k, \gamma_n]$. Then p divides the degree of the irreducible character χ_n of S_n associated with α_n.*

Proof. Let $p = k+2i$ be a prime in $(k, \gamma_n]$. It is well-known that the power to which p appears in $n!$ is $\sum_{m=1}^{\infty} \lfloor n/p^m \rfloor$, which equals $\lfloor n/p \rfloor$ since p is a middle prime. As observed in the introduction, by looking at the values of γ_n, we see that the only multiples of p that could occur as hook lengths are

p and $2p$. Thus, once we show $NH(n,p)+NH(n,2p)$, the total number of such hook lengths, is less than $\lfloor n/p \rfloor$ it follows that p divides the degree of χ_n. We obtain that inequality by proving $p(NH(n,p)+NH(n,2p)+1) \leq n$.

Case 1. Suppose $2p$ is a hook length.

A trivial calculation shows $p \leq \gamma_n/2 \leq 2k-1$ in all cases. This allows us to exploit the strict inequalities in Lemma 5 when appropriate.

First we consider $n = k^2 + j$ where $0 \leq j \leq k$.

We read the bounds from Lemmas 5 and 6 and utilize strict inequalities, when possible, to calculate that $NH(n,p)+NH(n,2p)$ is at most

$\left(\frac{3k+1}{4} - i\right) + \left(\frac{k+3}{4} - 2i - 1\right) = k - 3i$ when $j = 0$ and $k \equiv 1 \mod 4$;

$\left(\frac{3k+1}{4} - i - 1\right) + \left(\frac{k+3}{4} - 2i\right) = k - 3i$ when $0 < j < k$ and $k \equiv 1 \mod 4$;

$\left(\frac{3k+3}{4} - i - 1\right) + \left(\frac{k+1}{4} - 2i\right) = k - 3i$ when $j < k$ and $k \equiv 3 \mod 4$;

$\left(\frac{3k+1}{4} - i\right) + \left(\frac{k+3}{4} - 2i\right) = k - 3i + 1$ when $j = k$ and $k \equiv 1 \mod 4$;

$\left(\frac{3k+3}{4} - i\right) + \left(\frac{k+1}{4} - 2i\right) = k - 3i + 1$ when $j = k$ and $k \equiv 3 \mod 4$.

In the first three cases, $p(NH(n,p)+NH(n,2p)+1) = (k+2i)(k-3i+1) \leq k^2 - 4 < n$ by Lemma 2. In the remaining cases, $n = k^2 + k$ and we again use Lemma 2 and obtain $(k+2i)(k-3i+2) \leq k^2 + k - 2 = n - 2 < n$.

Now turn to $n = k^2 + k + j$ with $1 \leq j \leq k$. Using lemmas 5 and 6, $NH(n,p)+NH(n,2p)$ is at most

$\left(\frac{3k+5}{4} - i - 1\right) + \left(\frac{k+3}{4} - 2i\right) = k - 3i + 1$ when $j < k$ and $k \equiv 1 \mod 4$;

$\left(\frac{3k+3}{4} - i - 1\right) + \left(\frac{k+5}{4} - 2i\right) = k - 3i + 1$ when $j < k$ and $k \equiv 3 \mod 4$;

$\left(\frac{3k+5}{4} - i\right) + \left(\frac{k+3}{4} - 2i\right) = k - 3i + 2$ when $j = k$ and $k \equiv 1 \mod 4$;

$\left(\frac{3k+3}{4} - i\right) + \left(\frac{k+5}{4} - 2i\right) = k - 3i + 2$ when $j = k$ and $k \equiv 3 \mod 4$.

By Lemma 2, $(k+2i)(k-3i+2) \leq k^2+k-2 < n$ and $(k+2i)(k-3i+3) \leq k^2 + 2k = n$ when $j = k$.

Case 2. Suppose $2p$ is not a hook length.

Here we have $NH(n,2p) = 0$, so we seek to show $p(NH(n,p)+1) \leq n$.

Suppose $k \equiv 3 \mod 4$. By Lemmas 5 and 3, $p(NH(n,p)+1) \leq (k+2i)(\frac{3k+3}{4} - i + 1) \leq \frac{25k^2+70k+49}{32}$. If $k \geq 11$, this last expression is clearly less than or equal to k^2 and so less than or equal to n. If $k = 7$, that expression is less than 56 which is at most n except when $n = 49 + j$, $0 \leq j < 7$. In those exceptional cases, the middle primes are 11, 13, and 17. By Lemma 5, noting the strict inequality for 11 and 13, $NH(n,11) \leq (3 \cdot 7 + 3)/4 - 2 - 1 = 3$, $NH(n,13) \leq 2$ and $NH(n,17) \leq 1$. Hence, $p(NH(p,n)+1) \leq n$ in these cases as well.

Suppose $k \equiv 1 \mod 4$. When $n = k^2 + j$, $0 \le j \le k$, $p(NH(n,p)+1) \le (k+2i)(\frac{3k+1}{4} - i + 1) \le \frac{25k^2+50k+25}{32} \le k^2 \le n$ by Lemmas 5 and 3 and the fact that $k \ge 7$. If $n = k^2 + k + j$, $1 \le j \le k$, then $p(NH(n,p)+1) \le (k+2i)(\frac{3k+5}{4} - i + 1) \le \frac{25k^2+90k+81}{32} \le k^2 + k + 1 \le n$ if $k \ge 13$. If $k = 9$, then $1 \le i \le 7$ and so arithmetic shows $(k+2i)(\frac{3k+5}{4} - i + 1) \le 91 \le n$.

$\square$

2.2 Middle Primes when k is Even

Lemma 7. *Suppose k is an even integer and $k \ge 7$. If $p = k + 2i - 1$ is an odd integer in $(k, \gamma_n]$, then we have the following values of γ_n and $LR(n,p)$, depending upon the residue class of $k \mod 4$. The integer j satisfies $0 \le j \le k/2 - 1$.*

$k \equiv 2 \mod 4$			
n	γ_n	*Parity of* γ_n	$LR(n,p)$
$k^2 + j$	$\frac{5k-4}{2}$	*odd*	$\frac{3k+2}{4} - i$
$k^2 + k/2 + j$	$\frac{5k-4}{2}$	*odd*	$\begin{cases} \frac{3k+6}{4} - i, & \text{if } p \le \frac{k+4}{2} + 2j \\ \frac{3k+2}{4} - i, & \text{if } p > \frac{k+4}{2} + 2j \end{cases}$
$k^2 + k + j$	$\frac{5k-2}{2}$	*even*	$\begin{cases} \frac{3k+6}{4} - i, & \text{if } p \le \frac{3k}{2} + 2j \\ \frac{3k+2}{4} - i, & \text{if } p > \frac{3k}{2} + 2j \end{cases}$
$k^2 + k + k/2 + j$	$\frac{5k-2}{2}$	*even*	$\frac{3k+6}{4} - i$
$k^2 + 2k$	$\frac{5k}{2}$	*odd*	$\frac{3k+6}{4} - i$

$k \equiv 0 \mod 4$			
n	γ_n	*Parity of* γ_n	$LR(n,p)$
k^2+j	$\frac{5k-4}{2}$	*even*	$\begin{cases} \frac{3k+4}{4}-i, & \textit{if } p \le \frac{3k-2}{2}+2j \\ \frac{3k}{4}-i, & \textit{if } p > \frac{3k-2}{2}+2j \end{cases}$
$k^2+k/2+j$	$\frac{5k-4}{2}$	*even*	$\dfrac{3k+4}{4}-i$
k^2+k+j	$\frac{5k-2}{2}$	*odd*	$\dfrac{3k+4}{4}-i$
$k^2+k+k/2+j$	$\frac{5k-2}{2}$	*odd*	$\begin{cases} \frac{3k+8}{4}-i, & \textit{if } p \le \frac{k+6}{2}+2j \\ \frac{3k+4}{4}-i, & \textit{if } p > \frac{k+6}{2}+2j \end{cases}$
k^2+2k	$\frac{5k}{2}$	*even*	$\begin{cases} \frac{3k+8}{4}-i, & \textit{if } p \le \frac{3k+2}{2} \\ \frac{3k+4}{4}-i, & \textit{if } p > \frac{3k+2}{2} \end{cases}$

Proof. The values and parities of γ_n may be calculated just as in the odd case.

Consider the case $n = k^2 + j$, $0 \le k \le k/2 - 1$. The first two columns of the hook graph for α_n are

$$\begin{array}{cc} \gamma_n & \gamma_n - 1 \\ \gamma_n - 2 & \gamma_n - 3 \\ \dots & \dots \\ \underline{\gamma_n - 2(k/2-j-1)} & \underline{\gamma_n - 2(k/2-j-1) - 1} \\ \gamma_n - 2(k/2-j-1) - 1 & \gamma_n - 2(k/2-j-1) - 2 \\ \dots & \dots \\ \gamma_n - 2(k-3) - 1 & \gamma_n - 2(k-3) - 2 \\ \gamma_n - 2(k-2) - 1 & \gamma_n - 2(k-2) - 2 \end{array}$$

For reference, row $k/2 - j$ is underlined. The hook lengths in rows 1 through $k/2-j$ of column 1 along with those in rows $k/2-j+1$ through k of column 2 have the same parity as γ_n while all the other hook lengths in those columns have the opposite parity.

Those two columns contain all the integers from γ_n to $\gamma_n - 2(k-2) - 2 = k/2$. Therefore, the lowest row of the hook graph containing p contains p in one of those first two columns.

Suppose $k \equiv 2 \mod 4$. In this case, the odd integers $\gamma_n - 2(m-1)$ occur in rows 1 to $k/2 - j$ of column 1 for $1 \le m \le k/2 - j$ and in rows $k/2 - j + 1$ to k of column 2 for $k/2 - j + 1 \le m \le k$. Thus, $LR(n,p) = m$ where

$p = \gamma_n - 2(m-1)$, which gives $m = \frac{3k+2}{4} - i$.

Next suppose $k \equiv 0 \mod 4$. Now the odd integers $\gamma_n - 2(m-1) - 1$ make their lowest appearance in rows 1 to $k/2 - j - 1$ of column 1 for $1 \le m < k/2 - j$ and in rows $k/2 - j + 1$ to k of column 2 for $k/2 - j \le m \le k$. Thus, $LR(n,p) = \begin{cases} m & \text{if } p > \gamma_n - 2(k/2 - j - 1) - 1 = \frac{3k-2}{2} + 2j \\ m+1 & \text{if } p \le \gamma_n - 2(k/2 - j - 1) - 1 = \frac{3k-2}{2} + 2j \end{cases}$

where $p = \gamma_n - 2(m-1) - 1$, yielding $m = \frac{3k}{4} - i$.

Turn now to the case $n = k^2 + k/2 + j$, $0 \le j \le k/2 - 1$. The calculations are similar to those in previous cases, so less detail is included. The first two columns of the hook graph are

$$\begin{array}{ccl}
\gamma_n & \gamma_n - 1 & \\
\gamma_n - 1 & \gamma_n - 2 & \\
\cdots & \cdots & \\
\underline{\gamma_n - 2(k-j-2) + 1} & \underline{\gamma_n - 2(k-j-2)} & \\
\gamma_n - 2(k-j-2) & \gamma_n - 2(k-j-2) - 1 & \text{(bottom row when } j = 0) \\
\cdots & \cdots & \\
\gamma_n - 2(k-3) & \gamma_n - 2(k-3) - 1 & \\
\gamma_n - 2(k-2) & \gamma_n - 2(k-2) - 1 &
\end{array}$$

For reference, row $k - j - 1$ is underlined. Notice the parity change in each column at rows 2 and $k - j$. Also note the lowest appearance of p will be in one of these columns.

Assume $k \equiv 2 \mod 4$. We see $LR(n,p) = \begin{cases} m & \text{if } p > \gamma_n - 2(k-j-2) = \frac{k+4}{2} + 2j \\ m+1 & \text{if } p \le \gamma_n - 2(k-j-2) = \frac{k+4}{2} + 2j \end{cases}$

where $p = \gamma_n - 2(m-1)$, which gives $m = \frac{3k+2}{4} - i$.

In case $k \equiv 0 \mod 4$, the lowest appearance of each odd integer $\gamma_n - 2(m-1) - 1$ for $1 \le m \le k-1$ is in row $m+1$. Solving for m as usual, $LR(n,p) = \frac{3k+4}{4} - i$.

Proceeding to the case $n_1 = k^2 + k + j$, $0 \le j \le k/2 - 1$, we let $n = k^2 + j$. By comparing the hook graph for α_{n_1} with that of α_n as in the proof of Lemma 4, we find $LR(n_1,p) = \begin{cases} LR(n,p) & \text{if } p > \gamma_n - 2(k/2 - j - 1) = \frac{3k}{2} + 2j \\ LR(n,p) + 1 & \text{if } p \le \gamma_n - 2(k/2 - j - 1) = \frac{3k}{2} + 2j \end{cases}$

when $k \equiv 2 \mod 4$ and

$$LR(n_1,p) = \begin{cases} LR(n,p)+1 & \text{if } p > \gamma_n - 2(k/2-j-1) - 1 = \frac{3k-2}{2} + 2j \\ LR(n,p) & \text{if } p \le \gamma_n - 2(k/2-j-1) - 1 = \frac{3k-2}{2} + 2j \end{cases}$$

when $k \equiv 0 \mod 4$, Substituting the values for $LR(n,p)$ from above completes the proof in this case.

The proofs for $n_1 = k^2 + k + k/2 + j$, $0 \le j \le k/2 - 1$, and $n_1 = k^2 + 2k$ may be done using similar methods. □

Lemma 8. *Suppose k is an even integer and $k \ge 8$. If $p = k + 2i - 1$ is an odd integer in $(k, \gamma_n]$, then we have the following upper bounds on $NH(n,p)$, depending upon the residue class of $k \mod 4$. In each case, $0 \le j \le k/2 - 1$.*

$k \equiv 2 \mod 4$		
n	*Bound*	*Strict when*
$k^2 + j$	$\frac{3k+2}{4} - i$	$p \in (k, 2k-3]$
$k^2 + k/2 + j$	$\frac{3k+2}{4} - i$	$p \in (k, 2k-7]$
$k^2 + k + j$	$\frac{3k+6}{4} - i$	$p \in (k, 2k-3]$
$k^2 + k + k/2 + j$	$\frac{3k+6}{4} - i$	$p \in (k, 2k-5]$
$k^2 + 2k$	$\frac{3k+6}{4} - i$	$p \in (k, 2k-3]$

$k \equiv 0 \mod 4$		
n	*Bound*	*Strict when*
$k^2 + j$	$\frac{3k+4}{4} - i$	$p \in (k, 2k-3]$
$k^2 + k/2 + j$	$\frac{3k+4}{4} - i$	$p \in (k, 2k-5]$
$k^2 + k + j$	$\frac{3k+4}{4} - i$	$p \in (k, 2k-3]$
$k^2 + k + k/2 + j$	$\frac{3k+4}{4} - i$, *if* $(n,p) \neq (79,13)$	$p \in (k, 2k-7]$
	5, *if* $(n,p) = (79,13)$	–
$k^2 + 2k$	$\frac{3k+4}{4} - i$	$p \in (k, 2k-5]$

Proof. We first consider the case $n = k^2 + k/2 + j$, $0 \le j \le k/2 - 1$. We know $NH(n,p) \le LR(n,p)$. The indicated bounds follow directly from Lemma 7 when $k \equiv 0 \mod 4$ or when $k \equiv 2 \mod 4$ with $p > \frac{k+4}{2} + 2j$.

Beginning with the hook length in row 2 column $k/2 + 2$, the hook graph for α_n has a $(k-2-j) \times (j+1)$ block of even integers shown below with its surrounding columns of odd integers:

$$\begin{array}{cccccc}
2k-3 & 2k-4 & 2k-6 & \dots & 2k-2j-4 & 2k-2j-7 \\
2k-5 & 2k-6 & 2k-8 & \dots & 2k-2j-6 & 2k-2j-9 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2j+5 & 2j+4 & 2j+2 & \dots & 4 & 1 \\
2j+3 & 2j+2 & 2j & \dots & 2 &
\end{array}$$

If $p \in (k, 2k-5]$, then p appears in the leftmost column of odd integers above since $2j+3 \le 2(k/2-1)+3 = k+1$. Now p does not appear in the row of the block above this row. In other words, there is a number r with $r < LR(n,p)$ such that p does not appear in row r of the hook graph. Therefore, $NH(n,p) \le LR(n,p)-1$. (The case where $j=0$ may seem to require special attention, but in fact does not.) That gives the desired bound for $k \equiv 2 \mod 4$ with $p \le \frac{k+4}{2} + 2j$ since $k \ge 10$ implies $\frac{k+4}{2} + 2j \le \frac{k+4}{2} + 2(k/2-1) = 3k/2 \le 2k-5$. Moreover, it proves the indicated bounds are strict if $k \equiv 0 \mod 4$ and $p \in (k, 2k-5]$ or if $k \equiv 2 \mod 4$, $p \in (k, 2k-5]$, and $p > \frac{k+4}{2} + 2j$.

To finish the proof, we need only show the bound is strict when $k \equiv 2 \mod 4$, $p \in (k, 2k-7]$, and $p \le \frac{k+4}{2} + 2j$. In that situation, solving for j we have $j \ge \frac{1}{2}(k + 2i - 1 - \frac{k+4}{2}) = \frac{k-6}{4} + i \ge 2$ since $k \ge 10$. Thus, there are at least 3 columns in the above block. The number p appears in the leftmost column of odd integers at least two rows down from the top and p does not appear in the two rows above this row. Therefore, $NH(n,p) \le LR(n,p) - 2 < \frac{3k+2}{4} - i$ as was to be shown. That completes the proof for $n = k^2 + k/2 + j$, $0 \le j \le k/2-1$.

Skip to the case $n_1 = k^2 + k + k/2 + j$, $0 \le j \le k/2-1$. Let $n = k^2 + k/2 + j$. The bounds follow from $NH(n_1,p) \le LR(n_1,p)$ and Lemma 7 when $k \equiv 2 \mod 4$ and when $k \equiv 0 \mod 4$ with $p > \frac{k+6}{2} + 2j$.

The hook graph for α_{n_1} contains the same block of even hook lengths as did that of α_n (now beginning with the hook length in row 2 column k/2+3). Thus, for $p \in (k, 2k-5]$, the same row number r of the hook graph does not contain p and $r < LR(n,p) \le LR(n_1,p)$ which implies $NR(n_1,p) \le LR(n_1,p) - 1$. That gives the desired bound for $k \equiv 0 \mod 4$ with $p \le \frac{k+6}{2} + 2j$ except possibly when $k=8$, $p=13$ and $j=3$, because in all other cases $p \in (k, 2k-5]$. From the hook graph for α_{79} we find $NH(79,13) = 5 = LR(79,13) \left(= \frac{3k+8}{4} - i\right)$.

In addition, $NR(n_1,p) \le LR(n_1,p)-1$ proves the bounds mentioned earlier are strict if $k \equiv 2 \mod 4$ with $p \in (k, 2k-5]$ or if $k \equiv 0 \mod 4$ with $p \in (k,2k-5]$ and $p > \frac{k+6}{2}+2j$. Finally, suppose $k \equiv 0 \mod 4$ with $p \in (k,2k-7]$ and $p \le \frac{k+6}{2}+2j$. Solving for j, $j \ge \frac{k-8}{2}+i$, which is at least 1 and is at least 2 unless $k=8$ and $i=1$. As a result, for $k>8$, $i>1$ or $j \ge 2$, the two rows identified earlier in the hook graph of α_n still do not contain p in the hookgraph of α_{n_1}. Therefore, $NH(n_1,p) \le LR(n_1,p)-2 = \frac{3k+8}{2}-i-2 < \frac{3k+4}{4}$. If $k=8$, $i=1$ and $j=1$, then $p=9$ and an examination of the hookgraph of α_{77} yields $NH(77,9) < 6 = \frac{3k+4}{4}-i$.

Turn next to the case $n = k^2+j$, $0 \le j \le k/2-1$. The bounds follow directly from Lemma 7.

Beginning with the hook length in row 1 column $k/2+1$, the hook graph for α_n contains a $(k/2-j) \times (k/2+j)$ block of even integers

$$\begin{array}{cccc} 2k-2 & 2k-4 & \dots & k-2j \\ 2k-4 & 2k-6 & \dots & k-2j-2 \\ \vdots & \vdots & \ddots & \vdots \\ k+2j & k+2j-2 & \dots & 2. \end{array}$$

Assuming $p \in (k, 2k-3]$, the $p-1$ and $p+1$ are among the consecutive even integers from $2k-2$ to 2 which appear across the first row and down the last column of the block. Arguing as in the proof of Lemma 5, the bound is strict in this case.

Moving onto the cases $n_1 = k^2+k+j$ with $0 \le j \le k/2-1$ or $n_1 = k^2+2k$, the bounds follow as before from Lemma 7. The strictness conditions may be deduced much as in the case $n_1 = k^2+k+k/2+j$ by using the block of even hook lengths above. $\square$

Lemma 9. *Suppose k is an even integer and $k \ge 8$. If $p = k+2i-1$ and $2p$ are integers in $(k, \gamma_n]$, then we have the following bounds on $NH(n,2p)$.*

	$k \equiv 2 \mod 4$	$k \equiv 0 \mod 4$
n	*Bound*	*Bound*
k^2+j, $0 \le j \le k$	$\frac{k+6}{4}-2i$	$\frac{k+4}{4}-2i$
k^2+k+j, $1 \le j \le k$	$\frac{k+6}{4}-2i$	$\frac{k+8}{4}-2i$

Proof. By repeated application of Lemma 1 we see $NH(n,2p) \le LR(n,2p)$ is at most $LR(k^2+k,2p)$ when $n = k^2+j$, $0 \le j \le k$ and is at most

$LR(k^2+2k, 2p)$ in all cases. For $n_1 = k^2+k$ or k^2+2k, the first two columns of the hook graph for α_{n_1} are

$$\begin{array}{cc}
\gamma_{n_1} & \gamma_{n_1}-1 \\
\gamma_{n_1}-2 & \gamma_{n_1}-3 \\
\cdots & \cdots \\
\gamma_{n_1}-2(k/2-1) & \gamma_{n_1}-2(k/2-1)-1 \\
\hline
\gamma_{n_1}-2(k/2-1)-1 & \gamma_{n_1}-2(k/2-1)-2 \\
\cdots & \cdots \\
\gamma_{n_1}-2(k-3)-1 & \gamma_{n_1}-2(k-3)-2 \\
\gamma_{n_1}-2(k-2)-1 & \gamma_{n_1}-2(k-2)-2
\end{array}$$

with row $k/2$ underlined. Since $p > k \geq 8$, the values of γ_{n_1} show $\gamma_{n_1} - 2(k/2-1)-1 \leq 3k/2-2(k/2-1)-1 < 2k < 2p$. Thus, $2p$ does not appear below row $k/2$.

Examining rows 1 to $k/2$, we see $LR(n_1, 2p) = m$ where $2p = \gamma_{n_1} - 2(m-1)-1$ when γ_{n_1} is odd, while $2p = \gamma_{n_1} - 2(m-1)$ when γ_{n_1} is even. Taking $n_1 = k^2+k$ when $k \equiv 0 \mod 4$ and $n = k^2+j$, $0 \leq j \leq k$, and taking $n_1 = k^2+2k$ otherwise, then solving for m give the bounds in each case. $\square$

Proposition 2. *Suppose that $k^2 \leq n < (k+1)^2$ with k an even integer, $k \geq 8$, and p is a prime in $(k, \gamma_n]$. Then p divides the degree of the irreducible character χ_n of S_n associated with α_n.*

Proof. As in the proof of Proposition 1, it suffices to show $p(NH(n,p) + NH(n,2p)+1) \leq n$.

Case 1. Suppose $2p$ is a hooklength.

In this case, we cannot have $k = 8$ because then $8 < p \leq \gamma_n/2 \leq 10$, contradicting the fact that p is a prime. In particular, the exceptional case $(n,p) = (79, 13)$ of Lemma 8 cannot occur. Moreover, with $k > 8$, it is easy to show $p \leq \gamma_n/2 \leq 2k-7$ which allows us to use the strict inequalities from Lemma 8.

Therefore, from Lemmas 8 and 9 we have $p(NH(n,p)+NH(n,2p)+1)$ is at most
$(\frac{3k+2}{4} - i - 1) + (\frac{k+6}{4} - 2i) = k-3i+1$ when $k \equiv 2 \mod 4$ and $k^2 \leq n < k^2+k$;
$(\frac{3k+4}{4} - i - 1) + (\frac{k+4}{4} - 2i) = k-3i+1$ when $k \equiv 0 \mod 4$ and $k^2 \leq n \leq k^2+k$;
$(\frac{3k+6}{4} - i - 1) + (\frac{k+6}{4} - 2i) = k-3i+2$ when $k \equiv 2 \mod 4$ and $k^2+k \leq n \leq k^2+2k$;

$(\frac{3k+4}{4} - i - 1) + (\frac{k+8}{4} - 2i) = k - 3i + 2$ when $k \equiv 0 \mod 4$ and $k^2 + k < n \leq k^2 + 2k$.

By Lemma 2, $p(NH(n,p)+NH(n,2p)+1)$ is at most $(k+2i-1)(k-3i+2) \leq k^2-1 < n$ in the first two cases and at most $(k+2i-1)(k-3i+3) \leq k^2+k \leq n$ in the last two cases.

Case 2. Suppose $2p$ is not a hooklength.

In this case, $NH(n, 2p) = 0$, so it suffices to show $p(NH(n,p) + 1) \leq n$.

First, assume $k = 8$. In the exceptional case, $(n,p) = (79, 13)$, $13(NH(79, 13)+1) = 13 \cdot 6 < 79$. Henceforth, we assume $(n,p) \neq (79, 13)$. We have $p \in (8, \gamma_n] \subseteq (8, 20]$, so $i = 2$, 3, 5 or 6 (recall p is prime). From Lemma 8, $p(NH(n,p) + 1) \leq (8 + 2i - 1)(\frac{3 \cdot 8+4}{4} - i + 1) = 66$, 65, 51 or 38, respectively. Thus, for $n \geq 66$ or $i = 5$ or $i = 6$, we are finished. When n is 64 or 65 and $i = 2$ or $i = 3$, one can check the result directly from the corresponding hook graphs.

Assume $k > 8$. From Lemmas 8 and 3, $p(NH(n,p) + 1)$ is at most
$(k + 2i - 1)(\frac{3k+2}{4} - i + 1) \leq \frac{25k^2+40k+16}{32} \leq k^2 \leq n$ when $k \equiv 2 \mod 4$ and $k^2 \leq n < k^2 + k$;
$(k + 2i - 1)(\frac{3k+6}{4} - i + 1) \leq \frac{25k^2+80k+64}{32} \leq k^2 + k \leq n$ when $k \equiv 2 \mod 4$ and $k^2 + k \leq n \leq k^2 + 2k$;
$(k + 2i - 1)(\frac{3k+4}{4} - i + 1) \leq \frac{25k^2+60k+36}{32} \leq k^2 \leq n$ when $k \equiv 0 \mod 4$, $k > 8$.
The penultimate inequality in each case follows easily from the fact that $k \geq 12$ in the last case and $k \geq 10$ in the others.

□

3 Small Primes

If m is a positive integer and p is a prime, $\nu_p(m)$ will denote the exponent to which p occurs in the factorization of m.

Lemma 10. *Let n be an integer with $n \geq 49$, and let p be a prime with $p \leq k$. Let a and b be the positive integers defined by $p^a \leq (5k+1)/2 < p^{a+1}$ and $p^b \leq n < p^{b+1}$. Assume $p \neq 5$ when $100 \leq n \leq 124$. Then $a < b$.*

Proof. Let c be the positive integer satisfying $p^c \leq k < p^{c+1}$. Clearly $b \geq 2c$. First let $p = 2$. Then

$$\frac{5 \cdot 2^c + 1}{2} \leq \frac{5k+1}{2} < \frac{5 \cdot 2^{c+1} + 1}{2} < 2^{c+3},$$

implying $a \le c+2$. If $c > 2$, then $b \ge 2c > c+2 \ge a$. We can assume that c is 1 or 2. Thus $k \le 7$. But since $n \ge 49$, it follows that $k = 7$. Then $(5k+1)/2 = 18$. Thus $a = 4$. Since $b \ge 5$, $b > a$.

Now assume p is an odd prime. Then

$$\frac{5 \cdot p^c + 1}{2} \le \frac{5k+1}{2} < \frac{5 \cdot p^{c+1} + 1}{2} < p^{c+2}.$$

Thus $a \le c+1$. Again since $b \ge 2c$, $b > a$ when $c \ge 2$. We can assume $c = 1$ and $a = 2$ (If $a = 1$, then $b > a$). Then $p^2 \le (5k+1)/2$ and $p \le \sqrt{(5k+1)/2}$. Now

$$\sqrt{\frac{5k+1}{2}} \le \frac{10k-2}{25}$$

if $k \ge 17$. Thus if $k \ge 17$,

$$p^3 \le \frac{5k+1}{2}\sqrt{\frac{5k+1}{2}} \le \frac{5k+1}{2} \cdot \frac{10k-2}{25} = k^2 - \frac{1}{25} < n.$$

In this case, $b \ge 3 > 2 = a$. We are left to consider $49 \le n \le 288$, $c = 1$ and $a = 2$. Since $7 \le k \le 16$, $18 \le (5k+1)/2 \le 40.5$. Because $a = 2$, we have $p = 3$ or $p = 5$.

With $p = 3$ and $c = 1$, $k \le 8$, and so $n \le 80$. If $49 \le n \le 80$, $7 \le k \le 8$, and $18 \le (5k+1)/2 \le 20.5$. Thus $a = 2$ and $b = 3$, giving $b > a$.

Consider now the case with $p = 5$. Since $7 \le k \le 16$, $18 \le (5k+1)/2 \le 40.5$. Since $a = 2$, $25 \le (5k+1)/2$, yielding $k \ge 9.8$, and so $k \ge 10$ because k is an integer. Thus $n \ge 100$. Since the integer range $[100, 124]$ is excluded when $p = 5$, $n \ge 125$. Then $b \ge 3 > 2 = a$. □

Lemma 11. *If $n \ge 49$ and p is a prime with $p \le k$, then p divides $\chi_n(1)$.*

Proof. The largest hook length γ_n in the diagram of $[\alpha_n]$ is one of $(5k-3)/2$, $(5k-1)/2$, or $(5k+1)/2$ when k is odd, and one of $(5k-4)/2$, $(5k-2)/2$, or $5k/2$ when k is even. Now let d, b, and a be the positive integers defined by $p^d \le \gamma_n < p^{d+1}$, $p^a \le (5k+1)/2 < p^{a+1}$, and $p^b \le n < p^{b+1}$, respectively. Clearly $d \le a$. Unless $100 \le n \le 124$ and $p = 5$, the Lemma 10 guarantees that $a < b$ and so $d < b$. When $100 \le n \le 119$, the largest hook length γ_n is 23 or 24. Thus when p is 5 in these cases, $d = 1$ and $b = 2$ giving $d < b$.

Assume now that $p \ne 5$ when $120 \le n \le 124$. For each positive integer i, let m_i be the number of hook lengths of the hook graph of the diagram

$[\alpha_n]$ divisible by p^i. Then by [3, Theorem 2.7.40], $m_i \leq \lfloor n/p^i \rfloor$ for every i. Denote the product of all the hook lengths of the hook graph by h. Now $\nu_p(h) = \sum_{i=1}^{\infty} m_i$. It is well-known that $\nu_p(n!)$ is $\sum_{i=1}^{\infty} \lfloor n/p^i \rfloor$. If $n \geq 49$ and $n \notin [120, 124]$ when $p = 5$, the above argument shows that $m_b = 0 < \lfloor n/p^b \rfloor$. Hence $\nu_p(h) < \nu_p(n!)$ and so p divides $\chi_{\alpha_n}(1) = n!/h$.

The cases $120 \leq n \leq 124$ and $p = 5$ may be checked directly by hand. $\square$

Note that the proofs of Lemmas 10 and 11 together show: if $n \geq 125$ and β_n is any partition of n with the property that the largest hook length of the hook graph of β_n equals γ_n, then $\chi_\beta(1)$ is divisible by all small primes.

4 Small n

To conclude the proof of Alvis' conjecture we must exhibit for each integer n in the interval $[15, 48]$ a partition whose associated character of S_n has the desired property—its degree is divisible by every prime less than or equal to n—and restricts irreducibly to A_n. We have exhibited partitions in Table 1 of [1] when $15 \leq n \leq 35$ already. Partitions which work when $36 \leq n \leq 48$ are:

36: (8,7,6,6,5,4)	37: (8,7,7,6,5,4)	38: (8,8,7,6,5,4)
39: (8,8,7,6,5,5)	40: (8,8,7,6,6,5)	41: (8,8,7,7,6,5)
42: (9,8,7,7,6,5)	43: (9,8,7,7,6,6)	44: (9,9,8,7,6,5)
45: (9,9,8,7,6,6)	46: (9,9,8,7,7,6)	47: (9,9,8,8,7,6)
48: (10,9,8,8,7,6).		

Notice that with the sole exception of 43 we have followed the earlier scheme for even k, in this case 6.

References

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[3] Gordon James and Adalbert Kerber, *The Representation Theory of the Symmetric Group*, Volume 16 in Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, 1981.

[4] Mark L. Lewis and Donald l. White, *Diameters of Degree Graphs of Nonsolvable Groups*, J. Alg. **283** No. 1 (2005), 80-92.