3 Equations and how to manipulate them

3.1 Introduction

The last chapter introduced many of the more common mathematical functions. It is essential that you know how to manipulate expressions containing combinations of such relationships. Sometimes this will be done in order to simplify the expressions. Sometimes it will be necessary to combine several expressions to produce a new one. Very often the form of an expression is inappropriate for a particular task. Whatever the reason, this is the chapter that tells you how to go about combining, simplifying and rearranging mathematical expressions.

Some of the equations that you will see in this chapter are unfamiliar geophysical or geochemical expressions. However, these will not be derived here because this is not a geophysics or geochemistry text. Enough will be said to allow you to understand the context of the problem.

3.2 Rearranging simple equations

It is very obvious, but it is vitally important to appreciate, that an equation is a mathematical statement in which two expressions equal one another. Look again at the lake bed sediment example from Chapter 1:

$$\text{Age} = k \times \text{Depth} \tag{1.1}$$

The left-hand expression is very simple, it contains 'age'. The right-hand side is also simple and is the product of k and 'Depth'. The point is that the left- and right-hand sides are stated to be equal and this is what makes 1.1 an equation. The reason that I labour this point is that the golden and unbreakable rule when manipulating equations is that, whatever you do, the left- and right-hand sides must remain equal to one another. This is simply achieved. Whenever you manipulate one side of an equation, you must perform exactly the same operation on the other side. Thus, if you add a constant to one side, you must add the same constant to the other side as well; if you double one side, you must double the other; and so on. For example, given Eqn. 1.1, the following expressions are also true:

$\text{Age} + 3 = (k \times \text{Depth}) + 3$ (i.e. add 3 to both sides);
$2 \times \text{Age} = 2k \times \text{Depth}$ (i.e. double both sides);

$\sqrt{\text{Age}} = \sqrt{(k \times \text{Depth})}$ (i.e. square root both sides);
$\log(\text{Age}) = \log(k \times \text{Depth})$ (i.e. take logarithms of both sides).

By combining suitable operations on the two sides of an equation, it is possible to rearrange an equation into another form. As an example, suppose that instead of an equation which tells us the age if we know the depth (i.e. Eqn. 1.1), we actually need an equation which tells us the depth we would need to dig, to reach sediments of a specified age. How do we do this? We must manipulate Eqn. 1.1 to give a new equation which has 'Depth =' on the left-hand side rather than 'Age ='. The problem is that 'Depth' only appears in combination with 'k'; it does not stand on its own. We must, somehow, remove 'k'. Now, if '$k \times$ Depth' is divided by 'k' then we are left with 'Depth'. However, if we do this to the right-hand side of Eqn. 1.1 we must also do this to the left-hand side. This gives

$$\text{Age}/k = \text{Depth}$$

which can obviously be rewritten as

$$\text{Depth} = \text{Age}/k \tag{3.1}$$

which is the expression that we wanted.

The above example is very simple and could probably have been done almost automatically by many readers of this book. However, it is important that you read very carefully through the logic of the above example.

> **Question 3.1** Manipulate Eqn. 1.1 to give an expression for k. If, at a depth of 3 m, the age is 3000 years, use your result to determine the sedimentation constant. (Assuming, of course, that Eqn. 1.1 is valid for the lake bed in question.)

As another example, what about manipulating the slightly more complex lake sediment expression from Chapter 2,

$$\text{Age} = (k \times \text{Depth}) + \text{Age of top} \tag{2.1}$$

to give an expression for depth? The problem is that there is another term on the right-hand side to remove. Should we remove k first or 'Age of top'? In fact, it does not really matter. If we try to remove 'k', by dividing by k as before, the result is

$$\text{Age}/k = \text{Depth} + (\text{Age of top}/k) \tag{3.2}$$

Note that both 'Age' and 'Age of top' now appear over k since all terms must be divided by k. An expression for the depth is then found by subtracting the second term on the right-hand side to give, after swapping left and right sides around, Depth = (Age/k) − (Age of top/k) (3.3)

Alternatively, we could have begun by attempting to remove 'Age of top' from the right-hand side of Eqn. 2.1. Subtracting 'Age of top' from Eqn. 2.1 gives

Age − Age of top = $k \times$ Depth (3.4)

Dividing this by k then yields

Depth = (Age − Age of top)/k (3.5)

which is the same as Eqn. 3.3 although written in a slightly different form. (If these look totally different to you, do not worry as it will be explained later in this chapter.)

> Question 3.2 Starting with Eqn. 2.1, derive an expression for the age of sediments at the surface of a dried-out lake bed. If the sedimentation constant was 5000 y m^{-1} and, at a depth of 10 m, the age was 60 000 years, determine the age of the surface sediments, this time assuming that Eqn. 2.1 is valid.

Yet another example: how do we know the mass of the Earth? The answer is that we know from the strength of gravity at the Earth's surface. The strength of gravity is measured by the acceleration it causes to a falling body; a strong gravitational pull will accelerate a falling apple, say, more than a weak gravitational pull. Physicists tell us that this gravitational acceleration, g, is related to the Earth's mass, M, by the equation

$$g = GM/r^2 \qquad (3.6)$$

where G is a known physical constant and r is the Earth's radius. To use this equation to estimate the Earth's mass it must be rearranged into an expression for M. Now, G and r^2 can be removed from the right-hand side by dividing by G and multiplying by r^2. Thus, the left-hand side must also be divided by G and multiplied by r^2 to give

$$gr^2/G = M$$

or, after swapping around

$$M = gr^2/G \qquad (3.7)$$

The values of all the symbols on the right-hand side of Eqn. 3.7 are known. The gravitational acceleration, g, and the gravitational constant G have both been measured very accurately in physicists' laboratories, whilst the radius of the Earth, r, has been known for more than 400 years (in fact, some ancient Greeks had a pretty good idea too!). The values are

$$g = 9.81 \text{ m s}^{-1}$$
$$r = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$$
$$G = 6.672 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

Do not worry too much about the units of these numbers but it is important that consistent units are used (Chapter 1), so I have converted the Earth's radius into metres since both g and G are given in units which include metres. Substituting these numbers into Eqn. 3.7 gives

$$\begin{aligned} M &= 9.81 \times (6.37 \times 10^6)^2/(6.672 \times 10^{-11}) \\ &= (9.81 \times 6.37^2/6.672) \times 10^{23} \\ &= 59.7 \times 10^{23} \\ &= 5.97 \times 10^{24} \text{ kg} \end{aligned}$$

which is about six thousand million million million tonnes. Not bad for a small planet!

3.3 Combining and simplifying equations

If we know the mass of the Earth and can also find its volume, the Earth's average density could be calculated. The volume of the Earth can be estimated using the standard formula for the volume, V, of a sphere of radius r. This is

$$V = 4\pi r^3/3 \qquad (3.8)$$

Using the radius of the Earth given above and $\pi \approx 3.142$ gives a volume of

$$\begin{aligned} V &= 4 \times 3.142 \times (6.37 \times 10^6)^3/3 \\ &= (4 \times 3.142 \times 6.37^3/3) \times 10^{18} \\ &= 1083 \times 10^{18} \\ &= 1.083 \times 10^{21} \text{ m}^3 \end{aligned}$$

The density (usually denoted by rho, ρ) is related to mass and volume by

$$\rho = M/V \qquad (3.9)$$

Thus, using the mass and volume already found, the average density of the Earth is given by

$$\begin{aligned} \rho &= (5.97 \times 10^{24})/(1.083 \times 10^{21}) \\ &= (5.97/1.08) \times 10^3 \\ &= 5.51 \times 10^3 \\ &= 5510 \text{ kg m}^{-3} \end{aligned}$$

and this is more than five times the density of water (which is around 1000 kg m^{-3}).

The above derivation was a little tortuous. It is possible to combine the three expressions (Eqns. 3.7–3.9) into a single expression for density. This means less numerical calculation is necessary and fewer errors will be made. Equation 3.9 tells us to divide mass by volume to give density. The mass is given by Eqn. 3.7 whilst the volume is given by Eqn. 3.8. Therefore we can immediately write down

$$\rho = M/V = \text{Eqn. 3.7/Eqn. 3.8}$$

$$= \frac{gr^2/G}{4\pi r^3/3} \quad (3.10)$$

In other words, it is always possible to replace an expression (e.g. M) by another equal expression (in this case gr^2/G). Since the initial and replacement expressions are equal, the right-hand side of the equation does not change its value and the equation remains true. We now have an expression for density which could be evaluated by substituting the known values for g, r and G. However, Eqn. 3.10 looks a bit daunting. Evaluating it is not any easier than separately evaluating M and V as before. Fortunately, it is possible to simplify. First, we multiply both the top of the right-hand side and the bottom of the right-hand side by G. Since we are multiplying the whole expression then by G/G (which equals 1.0), this has no effect upon the left-hand side. The result is

$$\rho = \frac{G(gr^2/G)}{G(4\pi r^3/3)}$$

$$= \frac{gr^2}{4G\pi r^3/3} \quad (3.11)$$

since the two Gs on the top cancel each other. Now we can do a similar trick to remove the division by 3. Multiplying top and bottom by 3 yields

$$\rho = \frac{3gr^2}{4G\pi r^3} \quad (3.12)$$

Finally, by noting that $r^3 = r^2 r$ and cancelling, this expression can be further reduced to

$$\rho = \frac{3gr^2}{4G\pi r^2 r}$$

$$= \frac{3g}{4G\pi r} \quad (3.13)$$

This is much simpler to use than Eqn. 3.10. Let's just check that it gives the right result:

$$\rho = 3g/(4G\pi r)$$
$$= 3 \times 9.81/(4 \times 6.672 \times 10^{-11} \times 3.142 \times 6.37 \times 10^6)$$
$$= [3 \times 9.81/(4 \times 6.672 \times 3.142 \times 6.37)] \times 10^5$$
$$= 0.0551 \times 10^5$$
$$= 5510 \text{ kg m}^{-3}$$

as before. Note that this result is about twice the density of typical rocks found at the Earth's surface. We can therefore conclude that the deep Earth must be much denser than the near surface for the average to come out so high.

> Question 3.3 Prove that if
> $w = 3y/(4z)$ and
> $x = 2y/(4z)$ then
> $w/x = 1.5$

These, then, are the basic tools for equation manipulation:
(i) you can add, multiply, divide, double, halve, subtract or perform any other operation you like, provided that you do exactly the same to both sides of an equation;
(ii) you can always replace an expression by any other expression which is equal to it.

3.4 Manipulating expressions containing brackets

An important mathematical skill is the ability to use brackets effectively. Sometimes an expression can be made a great deal easier to understand, and easier to use, if brackets are added or removed. Brackets will normally be added into an equation by a procedure called **factorization**, whilst the reverse operation which removes brackets is achieved by multiplying out.

We'll start with a simple problem which is analogous to the algebraic problem of multiplying out brackets. Imagine a region which is known to contain 2 tonnes of recoverable gold and 10 tonnes of recoverable silver in every square kilometre. How much gold and silver could be recovered from 2 km^2? Obviously it is 4 tonnes of gold and 20 tonnes of silver. In other words, you multiply the reserves in 1 km^2 by the number of km^2.

The following problem in bracket multiplication is identical to the problem above

$$2 \times (2x + 10y) = 4x + 20y \quad (3.14)$$

The left-hand side says that there are two lots of $(2x + 10y)$ and the right-hand side says that this is the same as $4x$ and $20y$. If you think of x as gold and y as silver you should see the equivalence of the two problems. In practice, all you do is multiply each of the terms inside the bracket by the number outside.

This rather easy example actually contains all the mathematics you need to tackle any other cases. For example,

$$2.3(x + 2y + 4z) = 2.3x + 4.6y + 9.2z \qquad (3.15)$$

is calculated exactly the same way as before. Simply multiply each of the terms inside the bracket by the number outside. If the number outside is a symbol rather than a number this leads to examples such as

$$a(x + 2y + 4z) = ax + 2ay + 4az \qquad (3.16)$$

which is no different from the earlier examples; simply multiply each term in the brackets by a. A final difficulty is if the number outside the bracket is a more complex expression such as $(a + 3)$ giving

$$(a + 3)(x + 2y + 4z) = (a + 3)x + 2(a + 3)y + 4(a + 3)z \qquad (3.17)$$

As you can see, this is still done the same way. However, this time we can take things a stage further. Each of the resulting terms in Eqn. 3.17 is a new problem in multiplying out brackets. For example, the first term on the right-hand side is

$$\begin{aligned}(a + 3)x &= x(a + 3)\\ &= ax + 3x\end{aligned} \qquad (3.18)$$

The other terms in Eqn. 3.17 can be similarly multiplied out leading to a final answer of

$$(a + 3)x + 2(a + 3)y + 4(a + 3)z = ax + 3x + 2ay + 6y + 4az + 12z \qquad (3.19)$$

Question 3.4 Multiply out the brackets in the following examples.
(i) $5(x + 2y)$;
(ii) $5(x + 2.2y)$;
(iii) $5.5(x + 2y)$;
(iv) $5a(x + 2y)$;
(v) $(x - 2y)(x + 2y)$;
(vi) $(x + 2y)^2$

Question 3.5 Earlier in this chapter I stated that the expression

$$\text{Depth} = (\text{Age}/k) - (\text{Age of top}/k) \qquad (3.3)$$

was exactly the same as

$$\text{Depth} = (\text{Age} - \text{Age of top})/k \qquad (3.5)$$

Verify this by rewriting Eqn. 3.5 in the form

$$\text{Depth} = (1/k)(\text{Age} - \text{Age of top})$$

and multiplying out the bracket.

Factorization is the reverse process to multiplying out of brackets. For example, Eqn. 3.18 above was

$$x(a + 3) = ax + 3x \qquad (3.18)$$

Factorization is the process of writing this the other way around:

$$ax + 3x = x(a + 3) \qquad (3.20)$$

Its main use is for simplifying the appearance of more complex expressions and relies upon spotting common factors. The trick is to spot that both terms on the left-hand side of Eqn. 3.20 contain a factor x, i.e. they are both equal to some quantity multiplied by x (a times x for the first term and three times x for the second). This x can be taken out as a common factor leaving $a + 3$ inside the bracket. Another, more difficult, example might be

$$3.2xy + 6.4xw + z = ? \qquad (3.21)$$

The first two terms have a common factor of $3.2x$ which can therefore be written outside a bracket containing $y + 2w$. Thus, the solution is

$$3.2xy + 6.4xw + z = 3.2x(y + 2w) + z \qquad (3.22)$$

Note that the third term does not have any factors in common with the first two and is therefore left alone.

Question 3.6 Factorize $6ax + 3ay$

Factorization can be used to derive an equation for the density of a wet, porous sandstone. This rock will be partly made from sand grains with a density of ρ_s and partly made from water with a density of ρ_w. Hence, the average density of the sample will be somewhere between ρ_s and ρ_w. If the porosity, ϕ, is low, the density will be close to that of the sand grains but if the porosity is higher, then the average density will be a little closer to that of water. More mathematically, take a specimen of this sandstone which has a volume V and mass m. This mass will be made up from the mass of water in the volume plus the mass of the grains, i.e.

$$m = m_w + m_s \qquad (3.23)$$

where m_w and m_s are the masses of water and sand respectively. However, the mass of water is given by the product of the volume of water and the density of water whilst m_s is similarly given by the grain density times the volume of grains. Thus, Eqn. 3.23 can be written

$$m = V_w\rho_w + V_s\rho_s \qquad (3.24)$$

where V_w and V_s are the volumes of water and sand in the sample. The volume of water is equal to the volume of the sample multiplied by the porosity (e.g. porosity equal to 0.5 implies that half the total volume is water, a porosity of 0.25 implies one-quarter of the total volume is water). Thus

$$V_w = \phi V \quad (3.25)$$

The remaining volume must be sand and therefore

$$V_s = V - \phi V \quad (3.26)$$

Substituting these volumes into Eqn. 3.24 gives

$$m = \phi V \rho_w + (V - \phi V)\rho_s \quad (3.27)$$

Multiplying out the bracket then leads to

$$m = \phi V \rho_w + V\rho_s - \phi V \rho_s \quad (3.28)$$

Now V is common to all the terms on the right-hand side and can therefore be taken out as a common factor to give

$$m = V(\phi\, \rho_w + \rho_s - \phi\, \rho_s) \quad (3.29)$$

Finally, dividing by V and performing a further factorization gives the required average density

$$\begin{aligned} \rho &= m/V \\ &= \phi\, \rho_w + \rho_s - \phi\, \rho_s \\ &= \phi\, \rho_w + (1 - \phi)\rho_s \end{aligned} \quad (3.30)$$

> **Question 3.7** Using Eqn. 3.30, plot a graph of how the density varies as porosity changes from zero to one. Assume
>
> $\rho_w = 1000$ kg m^{-3}
> $\rho_s = 2500$ kg m^{-3}

3.5 'Rearranging' quadratic equations

Chapter 2 introduced a quadratic expression for calculating temperature for the deeper parts of the Earth. This was in the form

$$\text{Temperature} = (-8.255 \times 10^{-5})z^2 + 1.05z + 1110 \quad (2.6)$$

How can this be rearranged to allow calculation of the depth for a given temperature, e.g. at what depth is the temperature 2000°C? In fact, such a rearrangement is rather difficult. To solve this problem it is first necessary to discuss a technique called finding the roots of a quadratic equation.

The roots of a quadratic equation are the values of the variable which make the quadratic expression equal zero. Figure 3.1 should make this idea clearer.

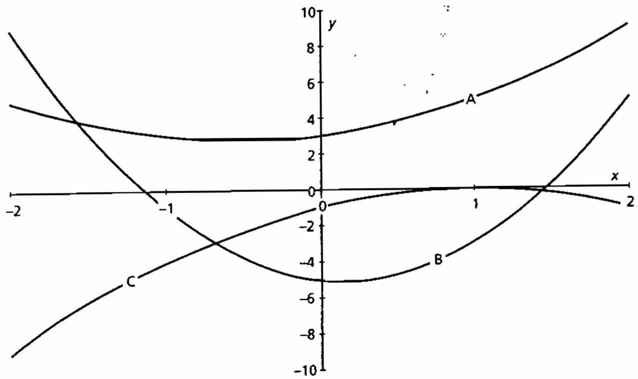

Fig. 3.1 The roots of a quadratic equation are the points where the quadratic curve crosses the horizontal axis. Thus, curve A has no roots, curve B has roots at about $x = -1.1$ and $x = 1.5$, curve C has one root near $x = 1$.

The most general way to find these roots is to use the following method. The roots of the quadratic equation

$$y = ax^2 + bx + c \quad (3.31)$$

are the values of x for which

$$y = 0 \quad (3.32)$$

or, equivalently, the roots are the values of x satisfying

$$0 = ax^2 + bx + c \quad (3.33)$$

From Fig. 3.1 it should be clear that there can be two such values (curve B), or one (curve C) or none (curve A). The values are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (3.34)$$

where ± means 'either add or subtract'. For example, curve B in Fig. 3.1 had the equation

$$y = 3x^2 - x - 5 \quad (3.35)$$

i.e. $a = 3, b = -1, c = -5$.

Note that both b and c are negative in this example. Substituting these values into Eqn. 3.34 gives

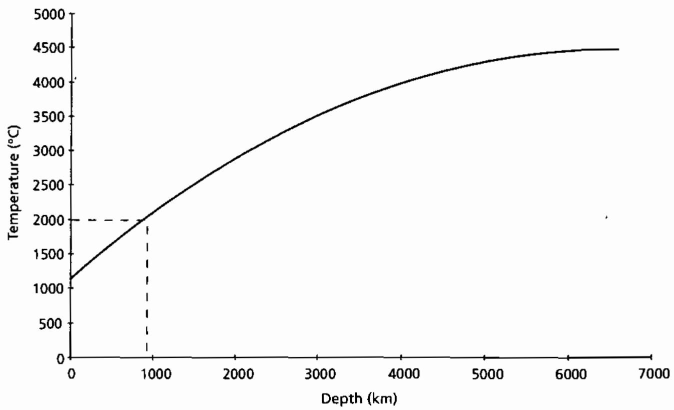

Fig. 3.2 At what depth is the temperature 2000°C?

$$x = [1 \pm \sqrt{}(1 + 4 \times 3 \times 5)]/[2 \times 3]$$
$$= [1 \pm \sqrt{}61]/6$$
$$= [1 + \sqrt{}61]/6 \text{ or } = [1 - \sqrt{}61]/6$$
$$= 1.47 \text{ or } = -1.14 \tag{3.36}$$

which are, indeed, the points where curve B crosses the horizontal axis in Fig. 3.1. If, on the other hand, we look at curve A, this has the equation

$$y = x^2 + x + 3 \tag{3.37}$$

i.e. $a = 1, b = 1, c = 3$.

Substituting these values into Eqn. 3.34 gives

$$x = [-1 + \sqrt{}(1 - 12)]/2$$
$$= [-1 \pm \sqrt{}-11]/2 \tag{3.38}$$

but $\sqrt{}-11$ has no solution (because you cannot find the square root of a negative number) and therefore there are no roots (which is what Fig. 3.1 shows).

> Question 3.8 Find the roots for curve C where
> $y = -x^2 + 2x - 1$

We can now attempt the problem of how to find the depth for a particular temperature. Take the case where we wish to know the depth at which the temperature is 2000°C (Fig. 3.2). Substituting this temperature into Eqn. 2.6 gives

$$2000 = (-8.255 \times 10^{-5})z^2 + 1.05z + 1110 \tag{3.39}$$

which is not quite in the form that we need. We have a method for solving equations like 3.33 where the quadratic expression equals zero. Here we have an equation which equals 2000. However, this is simply remedied by subtracting 2000 from both sides to give

$$0 = (-8.255 \times 10^{-5})z^2 + 1.05z - 890 \tag{3.40}$$

which is equivalent to Eqn. 3.33 with values for a, b and c of

$a = -8.255 \times 10^{-5}, b = 1.05, c = -890.$

Substituting these into Eqn. 3.34 (and remembering that z is the variable in Eqn. 2.6 not x) gives

$$z = \frac{-1.05 \pm \sqrt{1.05^2 - (4 \times 8.255 \times 10^{-5} \times 890)}}{-2 \times 8.255 \times 10^{-5}} \tag{3.41}$$

which has solutions of $z = 913$ km and $z = 11\,806$ km. There are two depths because at a depth of 11 806 km, you are about 913 km from the surface on the far side of the Earth.

> Question 3.9 Find the depths at which temperature reaches 3000°C.

3.6 Further questions

3.10 The density, ρ, of an air-filled porous rock is given by

$$\rho = \rho_g[1 - (V_p/V)]$$

where ρ_g is the density of the grains making up the rock, V_p is the volume occupied by pore space and V is the total volume. By combining this with Eqn. 3.9 prove that the grain density is given by

$$\rho_g = M/(V - V_p)$$

where M is the mass of the rock sample. Hence, calculate both the average density and the grain density of a sample with a volume of 0.11 m^3, a mass of 205 kg and a porosity of 0.32.

3.11 Stokes' law states that the velocity at which a spherical particle suspended in a fluid settles to the bottom is given by

$$v = \frac{2(\rho_p - \rho_f)gr^2}{9\eta}$$

where v is the velocity of descent, ρ_p and ρ_f are the densities of particle and fluid, respectively, g is the acceleration due to gravity, r is the particle radius and η is a property of the fluid known as its viscosity. Assuming that grains of different sizes have identical densities, show that the ratio of the settling velocities for two different grain sizes is

$$\frac{v_1}{v_2} = \left(\frac{r_1}{r_2}\right)^2$$

where v_1 and v_2 are the velocities for grains of radius r_1 and r_2, respectively. If a grain of radius 0.1 mm, suspended in a lake, takes 10 days to settle to the lake bottom, how long would it take a grain of radius 1 mm?

3.12 (i) Rearrange

$$0 = ax^2 + bx + c$$

into an equation for b.
(ii) Use your answer from (i) to verify that

$$b^2 - 4ac = a^2x^2 + (c^2/x^2) - 2ac$$

(iii) Verify that the answer to (ii) could be factorized to yield

$$b^2 - 4ac = [ax - (c/x)]^2$$

(Hint: It is easiest to do this by multiplying out the above expression.)
(iv) Use the above answers to verify that one of the roots of a quadratic expression is given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

3.13 Check your answers to questions 3.8 and 3.9 by using spreadsheet *Roots.xls*.

4 More advanced equation manipulation

4.1 Introduction

In the last chapter you were introduced to methods for manipulating simple equations. In this chapter we will look at a few, more advanced, techniques for equation manipulation. In particular, I shall discuss manipulation of equations containing exponentials and logarithms. I will also look at the topic of simultaneous equations in which several equations must be manipulated at the same time in order to solve a problem.

This chapter also introduces techniques for checking equations for errors. They may have been wrong in the first place (e.g. due to a printing error) or you may have made mistakes during your manipulations. Either way, it is useful to be able to check that equations are reasonable.

4.2 Expressions involving exponentials and logarithms

In Chapter 2 we looked at expressions involving exponentials such as

$$\phi = \phi_0 \, e^{-z/\lambda} \qquad (2.17)$$

for variation in porosity, ϕ, with depth, z. We also looked at logarithms and it was stated that these could be used to recast the graph of an exponential expression into the form of a straight line. This procedure is essential if, for example, you wish to rewrite Eqn. 2.17 to give the depth at which a particular porosity occurs. A little further revision on the properties of logarithms is first needed. The point to remember about logarithms is that they are simply the reverse operation to raising to a power (Section 2.8). From this, it follows that

$$\log_y(y^x) = x \qquad (4.1)$$

where y is any base for the logarithm.

For example, $10^2 = 100$ and $\log(100) = 2$, i.e., $\log(10^2) = 2$. Another useful result is that

$$\log(ab) = \log(a) + \log(b) \qquad (4.2)$$

where again the log can be to any base. In other words, the logarithm of any two numbers multiplied together is equal to the sum of the logarithms of the numbers. For example,

$$\log(12) = \log(3) + \log(4) \qquad \text{(since } 3 \times 4 = 12)$$

and also

$$\log(12) = \log(6) + \log(2) \qquad \text{(since } 2 \times 6 = 12)$$

or even

$$\log(12) = \log(10) + \log(1.2) \quad \text{(since } 10 \times 1.2 = 12)$$

Equation 4.2 can be generalized into

$$\log(abc\ldots) = \log(a) + \log(b) + \log(c) + \ldots \qquad (4.3)$$

i.e. the logarithm of a series of numbers multiplied together equals the sum of their logarithms. For example,

$$\ln(2 \times 3 \times 4.712 \times f) = \ln(2) + \ln(3) + \ln(4.712) + \ln(f)$$

A special case of Eqn. 4.3 is the logarithm of a number raised to a power. In this case we get

$$\log(x^n) = \log(x) + \log(x) + \log(x) + n \text{ terms like this}$$
$$= n\log(x) \qquad (4.4)$$

Finally, subtraction of two logarithms is equivalent to division of the arguments (the argument of $\log(b)$ is b, the argument of $\ln(f)$ is f and so on). Thus

$$\log(a/b) = \log(a) - \log(b) \qquad (4.5)$$

and

$$\ln(a/b) = \ln(a) - \ln(b) \qquad (4.6)$$

These rules allow the following transformation of equations such as 2.17.

Starting with

$$\phi = \phi_0\, e^{-z/\lambda} \qquad (2.17)$$

and taking the natural logarithm of both sides gives

$$\ln(\phi) = \ln(\phi_0\, e^{-z/\lambda}) \qquad (4.7)$$

Now, using the rule about logarithms of products (Eqn. 4.3), leads to

$$\ln(\phi) = \ln(\phi_0) + \ln(e^{-z/\lambda}) \qquad (4.8)$$

Finally, Eqn. 4.1 produces

$$\ln(\phi) = \ln(\phi_0) - z/\lambda \qquad (4.9)$$

which implies that a graph of $\ln(\phi)$ against z is a straight line of gradient $(-1/\lambda)$ and intercept $\ln(\phi_0)$ (Fig. 4.1).

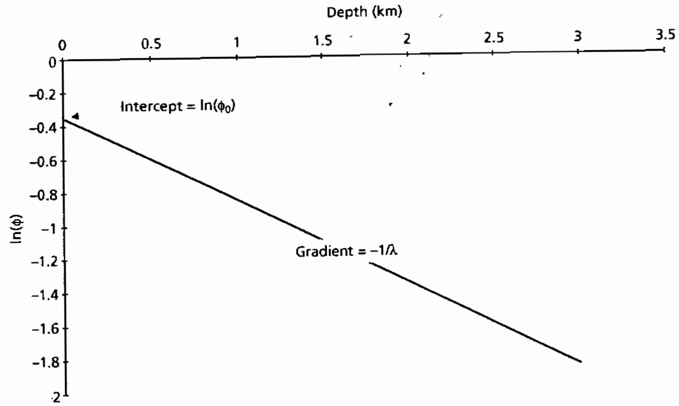

Fig. 4.1 A graph of ln(porosity) against depth assuming Eqn. 2.17, an initial porosity of 0.7 and $\lambda = 2$ km.

We now have an expression which can be rearranged for depth. First add z/λ to both sides of Eqn. 4.9 to give

$$z/\lambda + \ln(\phi) = \ln(\phi_0) \qquad (4.10)$$

Then subtract $\ln(\phi)$

$$z/\lambda = \ln(\phi_0) - \ln(\phi) \qquad (4.11)$$

Finally, multiply by λ

$$z = \lambda[\ln(\phi_0) - \ln(\phi)] \qquad (4.12)$$

Alternatively (using Eqn. 4.6), this can be expressed as

$$z = \lambda \ln(\phi_0/\phi) \qquad (4.13)$$

Thus, Eqn. 4.12 or 4.13 can now be used to obtain the depth at which a specific porosity occurs. For example, for $\lambda = 2$ km and $\phi_0 = 0.7$ a porosity of 0.35 would occur at a depth of

$$z = 2\ln(0.7/0.35)$$
$$= 2\ln(2)$$
$$= 2 \times 0.693$$
$$= 1.39 \text{ km}$$

A geochemical example may serve to reinforce these ideas. Radioactive dating is based upon the fact that the amount of a radioactive material decreases with time. Thus, if you know how much of the radioactive substance is