# 2 Common relationships between geological variables

## 2.1 Introduction

This chapter is about relationships between variables. In the last chapter, the depth and age of sediments in a lake were related by the simple expression

$$\text{Age} = k \times \text{Depth} \tag{1.1}$$

Many other geological variables are also related to each other. For example, the internal temperature of the Earth is related to depth (it gets hotter as you get deeper) and the strength of a rock is related to the pressure applied to it (rocks usually become stronger when compressed). However, the precise nature of the relationship varies from one example to another. For simple cases, expressions similar to Eqn. 1.1 will do. In other cases the relationships are more complex.

In Eqn. 1.1 above, age is a **function** of depth. This implies that any given depth produces a unique value for the age. Any type of relationship in which the value of one variable produces a single, unique, value for another is called a function and this term will be met with repeatedly throughout the remainder of this book.

This chapter is probably the most important in the entire book. It is only the fact that we can use mathematical expressions to relate different geological quantities that makes mathematics useful in geology. In practice, the true relationships between quantities such as depth and temperature are usually so complex that they must be approximated by much simpler ones. This chapter will introduce you to some of the most common of these simple relationships starting with the most simple and common of all, the straight line function.

## 2.2 The straight line

The straight line equation is possibly the most important mathematical expression found in geology since a very large range of geological problems can be approximated using straight line functions.

Returning to the lake sediment problem of Chapter 1, imagine that the lake completely dried out 1 My ago and that there has been no sedimentation in the lake since that time. Under these circumstances all the sediments are

| Depth (m) | Age (My) |
|---|---|
| 0 | 1 |
| 20 | 1.01 |
| 40 | 1.02 |
| 60 | 1.03 |
| 80 | 1.04 |
| 100 | 1.05 |

**Table 2.1** Age of sediments in a brief out take calculated using Eqn. 2.2.

1 My older than we might otherwise think, i.e. the top sediments are 1 My old rather than recent and sediments at a depth of 1 m are, say, 1 000 500 years old rather than 500 years old. An equation to describe the age of the sediments would now be

$$\text{Age} = (k \times \text{Depth}) + \text{Age of top} \quad (2.1)$$

since, in this expression, each age calculated from '$k \times$ Depth' has the 'Age of top' added to it. Thus, if the sedimentation constant $k$ was 500 y/m and the age of the top sediments is 1 My the sediments have an age of

$$\text{Age} = (500 \times \text{Depth}) + 1\,000\,000 \quad (2.2)$$

Sediments buried at a depth of 100 m would then have an age of

$$\begin{aligned}\text{Age} &= (500 \times 100) + 1\,000\,000\\ &= 1\,050\,000\\ &= 1.05\ \text{My}\end{aligned}$$

Repeating this calculation for a range of depths from 0 to 100 m gives the values shown in Table 2.1 and plotted in Fig. 2.1.

**Question 2.1** Repeat the above calculation for a depth of 50 m.

As you can see, the resultant graph is a straight line. A straight line is completely specified by just two quantities. Firstly, the point where the line crosses the vertical axis tells us how high up the line is. Secondly, the steepness of the line. A different straight line will either cross the vertical axis at another place or it will be less (or more) steep.

The position where the line crosses the vertical axis is called the **intercept** and has a value of 1 My in the particular case of Fig. 2.1. It is essential, when determining this intercept, that the vertical axis crosses the horizontal axis at the point where the depth is zero. If the vertical axis is anywhere else, the age at which the plotted line crosses the axis will be different (Fig. 2.2). Thus, to specify 'how high up' the straight line is, it would be necessary to give both the intercept and the location of the vertical axis. To avoid this the intercept is

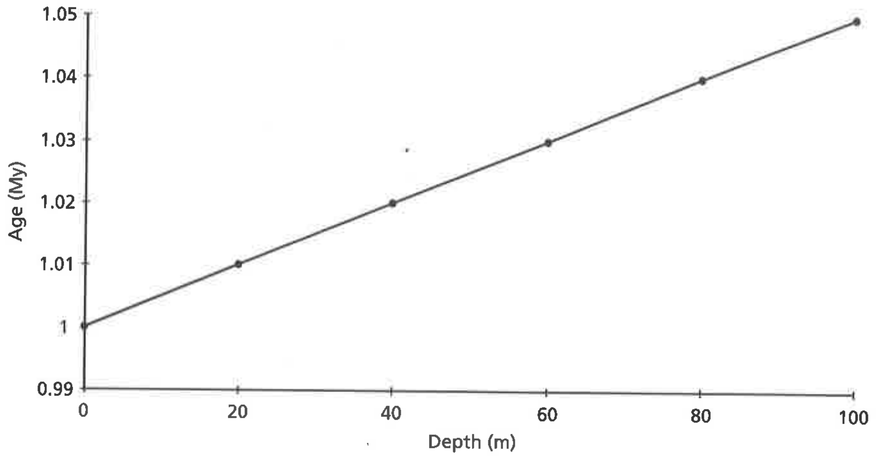


**Fig. 2.1** Graph of age versus depth data from Table 2.1.

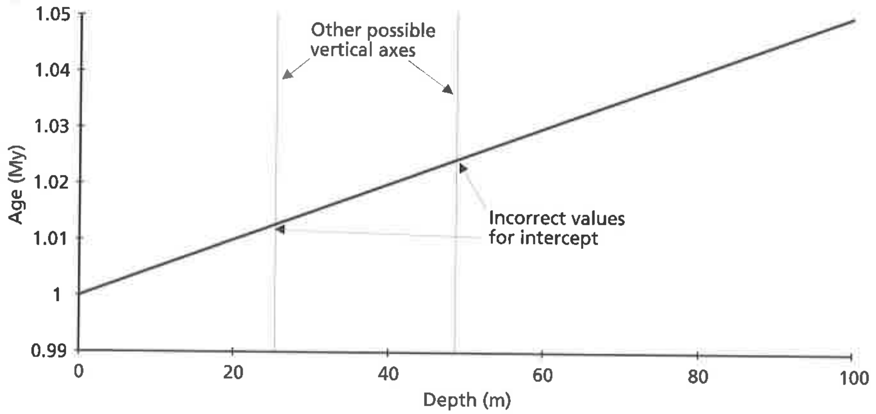


**Fig. 2.2** The intercept should be given for a vertical axis passing through zero on the horizontal axis. Any other vertical axis will give a different intercept.

always quoted for a vertical axis which passes through the origin of the horizontal axis.

The second value which characterizes a straight line, the steepness, is called the **gradient** of the line. This is simply a measure of how rapidly the 'height' increases as we move from left to right along the line. The effect of gradient is illustrated by Fig. 2.3. Note that both age and depth alter as we go from point A to point B and that, for a given change in depth, the change in age becomes greater as the steepness of the line increases. Thus, the steepness can be characterized by the increase in age produced by a given increase in depth. For simplicity, we can fix the depth increase as being 1 m, i.e. the gradient is defined as the increase in age produced by drilling into the sediments by an

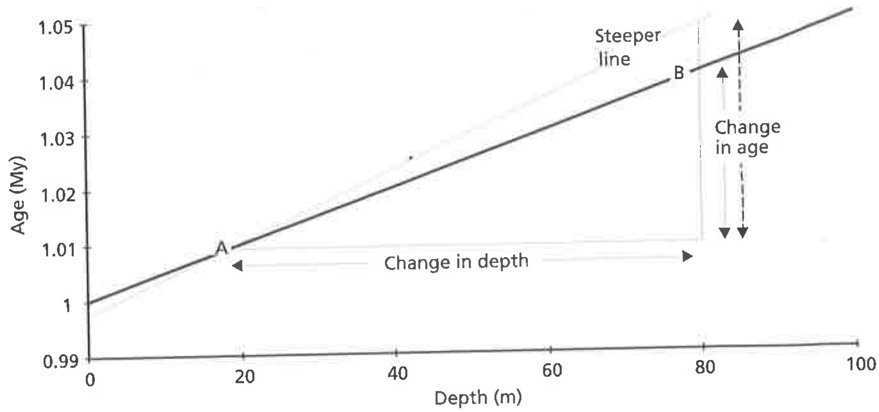


**Fig. 2.3** The age and depth alter as we go from point A to point B. However, the age alters more for the steeper line given the same change in depth.

additional 1 m. However, the points A and B in Fig. 2.3 are not necessarily 1 m apart so we must modify the observed change in age between points A and B by dividing it by the distance between them. Thus, the gradient is given by

Gradient = (Change in Age)/(Change in depth) (2.3)

For example, point A is at a depth of 20 m and an age of 1.01 My whilst point B is at a depth of 80 m and an age of 1.04 My. Thus, the change in depth is 60 m and the change in age is 0.03 My (= 30 000 years) giving a gradient of

Gradient = 30 000/60 = 500 y $m^{-1}$

There are several points to note about this answer. It does not matter which two points are chosen, the same answer would result if, for example, depths of 0 and 50 m had been chosen for the points A and B. Secondly, the 'units' for the answer of 'years/metre' occur because the top line of the calculation is in years (30 000 years) and the bottom line is in metres (60 m). Hence the calculation involves years divided by metres giving an answer in years/metre. This procedure for finding the units of an answer will be covered in more detail in Chapter 3.

> **Question 2.2** Calculate the gradient of the straight line in Fig. 2.3 using the point A again (depth = 20 m, age = 1.01 My) and the point at a depth of 100 m and age of 1.05 My.

The answer of 500 y $m^{-1}$ is not only a measure of the steepness of the line. This gradient tells us that each metre of sediment takes 500 years to accumulate (i.e. 500 years per metre). This is, of course, our sedimentation constant.

**Table 2.2** Measured ages and depths for sediments in a lake bed.

| Depth (m) | Age (years) |
|---|---|
| 0.5 | 1 020 |
| 1.3 | 2 376 |
| 2.47 | 5 008 |
| 4.9 | 10 203 |
| 8.2 | 15 986 |

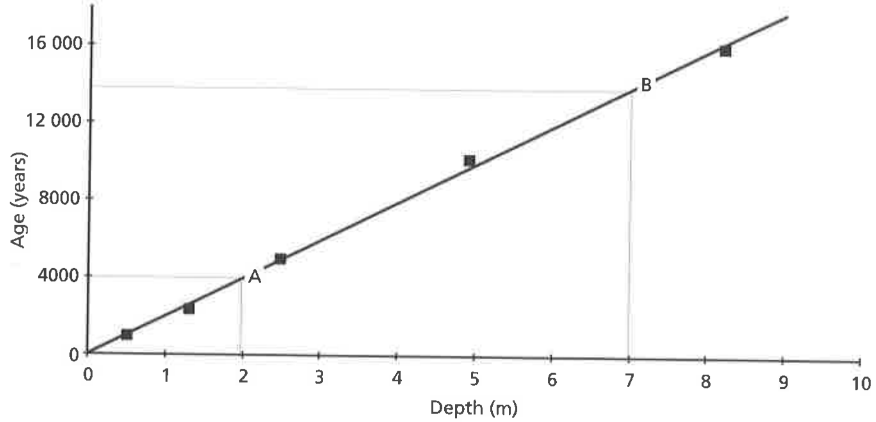


**Fig. 2.4** A graph showing the sediment age versus depth data from Table 2.2.

It should now be clear that, for the lake sedimentation example, the intercept on a graph of age against depth tells us the age of the top sediments whilst the gradient tells us the rate of accumulation. In fact, rather than obtaining the straight line graph from given values for intercept and gradient, it is more likely that these quantities will be estimated by fitting a straight line to a graph of some depth/age data. Consider the age versus depth data shown in Table 2.2. These figures might, for example, have been obtained by taking cores from a lake bottom and dating them using the radiocarbon method (this is a geochemical method for estimating the age of organic remains, the details of which are beyond the scope of this book). Figure 2.4 shows a graph of these data together with a 'best-fit' straight line which passes very close to all of the points.

In this example, the intercept value is not significantly different from zero. Thus, in this lake, sedimentation is continuing at the present day and at the same rate as in the past. The gradient of the line can be found by assessing the points A and B shown. The point A lies at a depth of 2 m and an age of 4000 years whilst the point B has a depth of 7 m and an age of 14 000 years. Thus, the change in depth is 5 m and the change in age is 10 000 years giving a gradient of

Gradient = 10 000/5 = 2000 y $m^{-1}$

i.e. each metre of sediment took 2000 years to accumulate or, equivalently, the sedimentation constant $k$ = 2000 y $m^{-1}$.

**Question 2.3** Given the following depth/age data from a dried-up lake bed, estimate the rate of sedimentation and how long ago the lake dried out.

| Depth (m) | Age (years) |
|---|---|
| 6 | 570 000 |
| 10 | 580 000 |
| 18 | 615 000 |
| 20 | 620 000 |

It is now time to move from the specific example of lake bottom sedimentation to more general expressions. If we start with our lake sediment equation,

$$\text{Age} = k \times \text{Depth} + \text{Age of top} \qquad (2.1)$$

this is not a general equation since the terms in Eqn. 2.1 (i.e. 'Age', '$k \times$ Depth', 'Age of top') refer to specific quantities involved in the sedimentation problem. The simplest way to arrive at a more general expression is to replace each of the terms in the equation by new symbols which do not have specific meanings. Thus we can replace the specific variable 'Age' by the general variable $y$. Similarly, the 'Depth' can be replaced by the general variable $x$ and the constants $k$ and 'Age of top' can be replaced by the general constants $m$ and $c$. This procedure results in a general form for the equation of a straight line of

$$y = mx + c \qquad (2.4)$$

where $y$ is plotted along the vertical axis and $x$ is plotted along the horizontal axis. Note that using the new symbols $y$, $m$, $x$ and $c$ was an entirely arbitrary choice. Any other set of symbols could have been chosen. For example, the equation

$$\alpha = \beta\gamma + \delta$$

is also a straight line equation, provided $\alpha$ and $\gamma$ are variables and $\beta$ and $\delta$ are constants, since it is of the same form as Eqns. 2.1 and 2.4. However, the particular symbols used in Eqn. 2.4 are traditionally used for the general form of the equation of a straight line and I have stuck to that convention.

A graph of Eqn. 2.4 is a straight line which has an intercept of $c$ and gradient of $m$ (Fig. 2.5). In this figure, $\Delta y$ (pronounced 'delta y', this means little bit

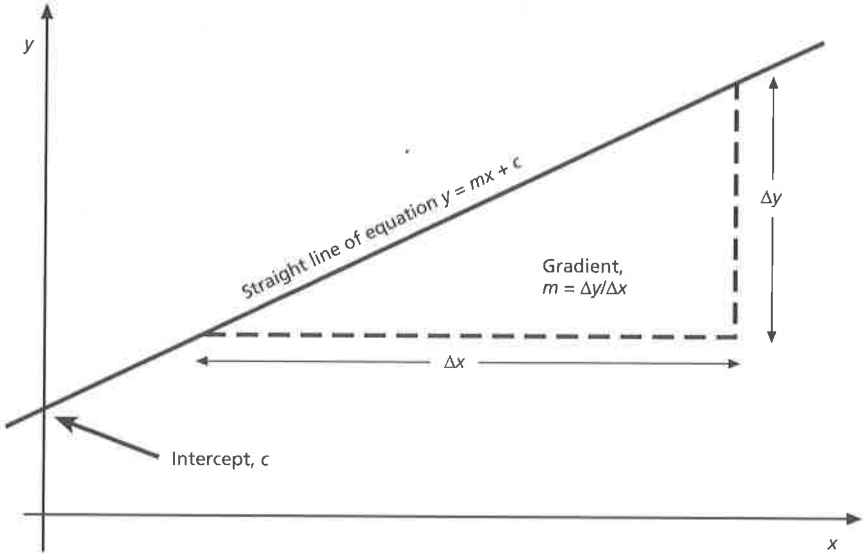


**Fig. 2.5** The general form of the equation of a straight line, $y = mx + c$.

of $y$) is the change in height produced by moving a horizontal distance $\Delta x$ along the line. Note that, in the illustrated example, $y$ increases for an increase in $x$ since the graph slopes up to the right. Thus $\Delta y$ and $\Delta x$ are positive and so is the gradient. On the other hand, if the line sloped downwards to the right, $y$ would decrease as $x$ increases. Thus, $\Delta y$ would be negative giving a negative gradient. As a general rule, lines that increase in height towards the right have a positive gradient whilst lines that decrease in height have negative gradients.

Spreadsheet *S_line.xls* should help to make this idea clearer. This sheet plots a straight line using Eqn. 2.4. You can alter the value of the gradient, $m$, and intercept, $c$, and observe the effect upon the resulting straight line.

The relationship between the general equation for a straight line (Eqn. 2.4) and specific real cases, should be made clearer by one last example. It is well known that temperature increases with depth in the Earth and, for depths of less than around 100 km, it is a good approximation to assume that a plot of temperature against depth should be a straight line. The intercept of such a graph is, by definition, the temperature at zero depth, i.e. the surface temperature. This value will vary considerably from tropical to polar locations but a typical value might be 10°C. The gradient of the line, i.e. the rate at which temperature increases with depth, also varies from one location to another since geologically active areas have very different gradients from old, stable, continental areas. However, values around 20°C $km^{-1}$ are not unusual, i.e. the temperature increases by 20°C for an increase in depth of

1 km. To summarize, temperature plotted against depth gives a straight line characterized by the local temperature gradient and an intercept equal to the local surface temperature. A general expression for how temperature varies with depth at a particular location would therefore be

$$\textit{Temperature} = (\textit{Gradient} \times \textit{Depth}) + \textit{Surface temperature} \tag{2.5}$$

(cf. $y = mx + c$). For the specific case of an intercept of 10°C and a gradient of 20°C km$^{-1}$ this would yield

$$\textit{Temperature} = (20 \times \textit{Depth}) + 10$$

Thus, at a depth of 40 km the temperature is 810°C.

**Question 2.4** Rocks usually increase in strength, $\tau$, when compressed. This strength is defined as the shearing (= sideways) pressure necessary for a particular rock specimen to break. The standard units of pressure are pascals. If $\tau$ increases by $m$ pascals for each additional pascal of normal (i.e. compressive pressure) and if the strength when not compressed is $\tau_0$, write an equation for how $\tau$ varies with normal pressure $P$. Sketch a graph of this function.

There are many other examples in geology of the use of straight line functions. However, many geological phenomena are not well represented by straight lines and more complex expressions must be used. Some of the more common alternatives are described in the remainder of this chapter.

## 2.3 Quadratic equations

The linear temperature with depth relationship discussed in the last section breaks down badly for depths much greater than 100 km. For example, at the centre of the Earth the depth is approximately 6360 km so that a surface temperature of 10°C and a gradient of 20°C km$^{-1}$ would predict a temperature of

$$\begin{aligned} \textit{Temperature} &= (\textit{Gradient} \times \textit{Depth}) + \textit{Surface temperature} \\ &= (20 \times 6360) + 10 \\ &= 127\,210°\text{C} \end{aligned} \tag{2.5}$$

In fact, the temperature in the Earth's core is only about 4300°C. Table 2.3 lists the approximate temperature in the Earth, at various depths, based upon geophysical and geochemical measurements.

The problem is that the temperature near the surface rises much more rapidly than it does deeper in the Earth, e.g. over 1000 degrees in the first 100 km but only 350 degrees in the following 300 km. Note that the temperature is virtually constant within the inner core (i.e. from 5100 km to the Earth's centre). Any attempt to extrapolate down to the core using the large

**Table 2.3** Temperature at various depths in the Earth as determined from geophysical measurements.

| Depth (km) | Temperature (°C) |
|---|---|
| 0 | 10 |
| 100 | 1150 |
| 400 | 1500 |
| 700 | 1900 |
| 2800 | 3700 |
| 5100 | 4300 |
| 6360 | 4300 |

rate of increase in temperature near the surface is bound to give a ridiculously large value. A much better approximation is

$$Temperature = (-8.255 \times 10^{-5})z^2 + 1.05z + 1110 \qquad (2.6)$$

where $z$ is the depth in kilometres. This equation contains three terms (i.e. $(-8.255 \times 10^{-5})z^2$, $1.05z$ and 1110) each of which is calculated separately before adding them together. Thus, at a depth $z = 5100$ km the temperature is

$$\begin{aligned} Temperature &= (-8.255 \times 10^{-5} \times 5100 \times 5100) + (1.05 \times 5100) + 1110 \\ &= -2147 + 5355 + 1110 = 4318°\text{C} \end{aligned}$$

which compares well with the value given in Table 2.3. However, at the Earth's surface, Eqn. 2.6 predicts

$$\begin{aligned} Temperature &= (-8.255 \times 10^{-5} \times 0 \times 0) + (1.05 \times 0) + 1110 \\ &= 1110°\text{C} \end{aligned}$$

which is certainly not correct. In fact, Eqn. 2.6 is a reasonable approximation to the Earth's internal temperature only for depths greater than around 100 km.

Figure 2.6 shows how the temperature variation predicted by Eqn. 2.6 compares with the temperatures given in Table 2.3. Although the fit is not exact, it is clear that Eqn. 2.6 can be used to calculate an approximate value for the temperature at any given depth below 100 km. Once again we see that mathematical descriptions of geological behaviour are useful approximations rather than exact 'truths'.

Equation 2.6 is a particular example of a **quadratic equation**. The general form is

$$y = ax^2 + bx + c \qquad (2.7)$$

where $y$ is a function of $x$ whilst $a$, $b$ and $c$ are constants. Figure 2.7 shows a selection of specific examples of Eqn. 2.7. Open spreadsheet *Quadrat.xls* to see, in more detail, how the values of $a$, $b$ and $c$ affect the shape of a quadratic function. Don't forget to try negative as well as positive values.

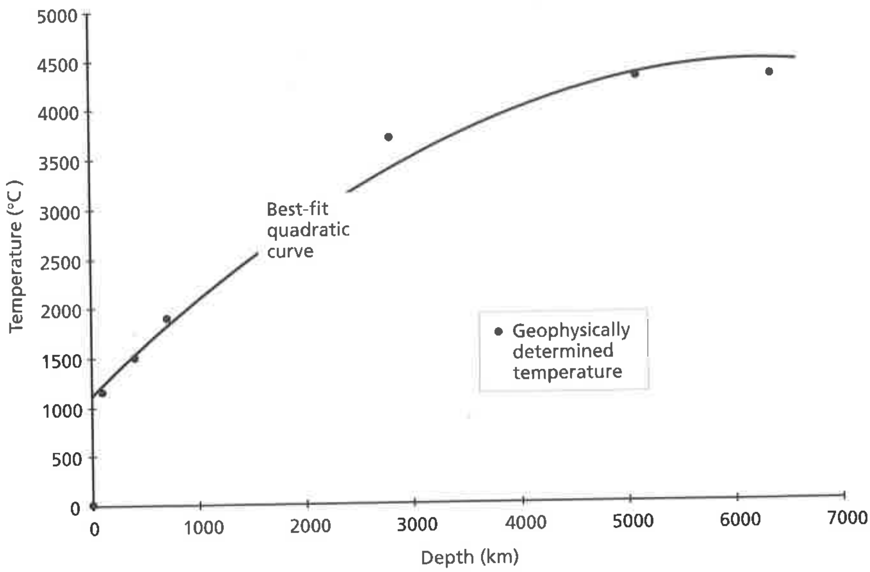


**Fig. 2.6** The temperature versus depth data from Table 2.3 compared to a best-fit quadratic function.

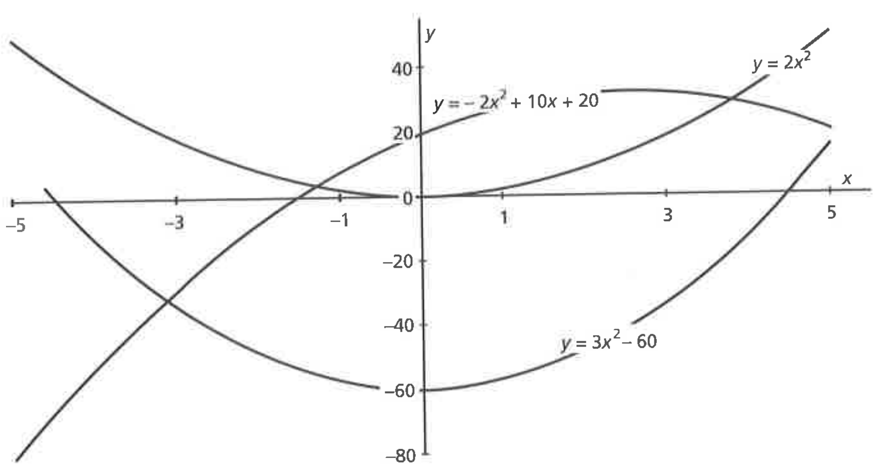


**Fig. 2.7** Examples of quadratic functions. Each curve has an equation of the form $y = ax^2 + bx + c$ but the values of $a$, $b$ and $c$ differ between the curves.

Comparing the general equation (Eqn. 2.7) with the temperature profile function (Eqn. 2.6) we can see that $y$ is equivalent to temperature and $x$ is equivalent to depth. In addition, the constants $a$, $b$ and $c$ have the values: $a = -8.255 \times 10^{-5}$; $b = 1.05$ and $c = 1110$. The ability to compare a particular equation to a standard form and pick out the appropriate values for the constants will be used again in this book, so make sure that you fully understand what has just been done.

**Question 2.5** If $f = 2g^2 - 10g + 6$
where $f$ and $g$ are variables, write down values for the constants equivalent to $a$, $b$ and $c$ in Eqn. 2.7.

## 2.4 Polynomial functions

It is possible to improve further on the fit of a mathematical expression to our temperature data by using longer expressions. Figure 2.8 compares the temperature data to the function

$$Temperature = az^4 + bz^3 + cz^2 + dz + e \qquad (2.8)$$

with values for the constants of $a = -1.12 \times 10^{-12}$, $b = 2.85 \times 10^{-8}$, $c = -0.000\,310$, $d = 1.64$ and $e = 930$. However, whilst the fit to the data is now much better, particularly between 2000 and 4000 km, the expression itself is becoming more difficult to evaluate. This trade-off between accuracy and simplicity is a frequent occurrence when applying mathematics to specific problems. Note that even this more complex expression does not model the temperature in the shallowest 100 km.

**Question 2.6** Compare the temperature predicted by Eqns. 2.6 and 2.8 at a depth of 2800 km. How do these results compare with the true value in Table 2.3?

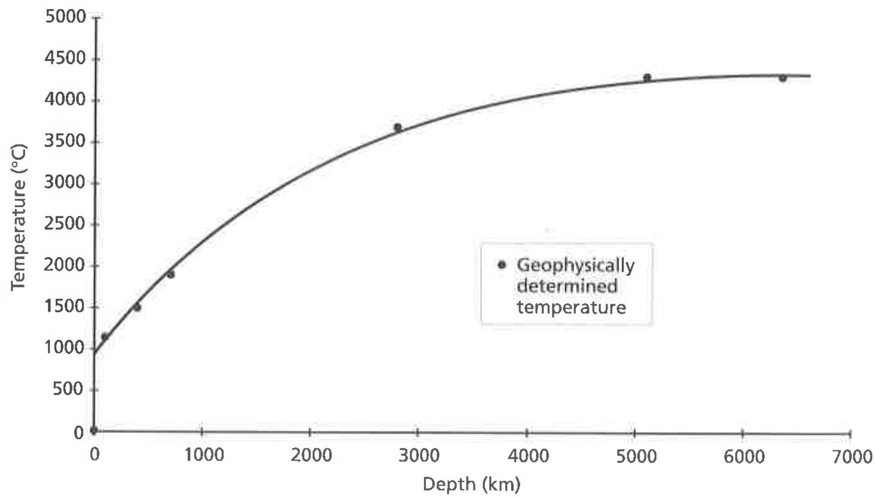


**Fig. 2.8** The temperature versus depth data from Table 2.3 compared to a best fit function of the form $Temperature = az^4 + bz^3 + cz^2 + dz + e$ where $z$ is depth.

Expressions like Eqn. 2.8 are known as polynomials (or power series). The general form for these is

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n \qquad (2.9)$$

in which '· · ·' indicates a numbér of terms which have not been explicitly written down. In this expression, $a_0$, $a_1$, $a_2$ etc. are constants and $n$ is an integer giving the power of the last term. For example, if $n = 2$, the expression simplifies to the quadratic function

$$y = a_0 + a_1x + a_2x^2$$

and, if $n = 1$, the expression is a straight line function

$$y = a_0 + a_1x$$

Thus, straight line functions and quadratic functions are special cases of polynomial functions. If $n = 5$, the resulting expression is

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

Spreadsheet *Poly.xls* allows you to investigate how the shape of this function depends upon the **coefficients** $a_0$, $a_1$ etc. Remember that you can set some of these coefficients to zero, e.g. set $a_2$, $a_3$, $a_4$ and $a_5$ all to zero to plot a straight line.

## 2.5 Negative powers

Look at Table 2.4 which shows $3^n$ for various values of $n$. For example, when $n = 1$, $3^n = 3^1 = 3$ and, when $n = 2$, $3^n = 3^2 = 9$. Now, starting at the top ($3^3 = 27$), and moving down the table, each successive result is 1/3 of the result above. For example, 9 is one-third of 27. Continuing this trend down the table, $3^0$ should be one-third of $3^1$, i.e. $3^0 = 1$. Taking the trend even further, $3^{-1}$ is 1/3 of 1 (i.e. $3^{-1} = 1/3$). A little further thought should convince you that the more general result is that $3^{-n} = 1/3^n$.

| $n$ | $3^n$ |
|---|---|
| 3 | 27 |
| 2 | 9 |
| 1 | 3 |
| 0 | 1 |
| −1 | 1/3 |
| −2 | 1/9 |
| −3 | 1/27 |

**Table 2.4** The result of raising 3 to the power of the integers between −3 and 3.

The most general result is that $x^n$ divided by $x$ is $x^{n-1}$. In other words, each decrease of the power by one is achieved by division by $x$. This is a special case of Eqn. 1.5,

$$x^a/x^b = x^{a-b} \quad (1.5)$$

in which $b = 1$. Thus, Eqn. 1.5 becomes

$$x^a/x = x^{a-1}$$

Now, $x/x = 1.0$ since any number divided by itself is 1.0. In addition, from the discussion above, $x/x = x^{1-1} = x^0$. Thus, $x^0 = 1.0$. This is a very important result: any number raised to the power of zero equals one. (The only exception is $0^0 = 0$.) For example, $2^0 = 1$, $100^0 = 1$, $(-36.4)^0 = 1$ and $\pi^0 = 1$.

This process can be taken a stage further by division of $x^0$ by $x$ to give $x^{-1}$. This is the same as dividing 1 by $x$, i.e. $x^{-1} = 1/x$. Further divisions lead to $x^{-2} = 1/x^2$, $x^{-3} = 1/x^3$, etc. In other words, a number raised to a negative power equals the reciprocal of the same number raised to a positive power. Thus, $(3.5)^{-96} = 1/(3.5)^{96}$, and, in general, $x^{-n} = 1/x^n$.

We are now in a position to see why, in the scientific notation introduced in Chapter 1, numbers smaller than one are expressed using a negative power of 10. Thus, 0.001 is written as $10^{-3}$ because it equals $1/10^3$ (= 1/1000).

## 2.6 Fractional powers

The last stage in the generalization of the use of powers is to allow the exponent to be a fraction. Table 2.4 and Fig. 2.9 should help to make this idea more

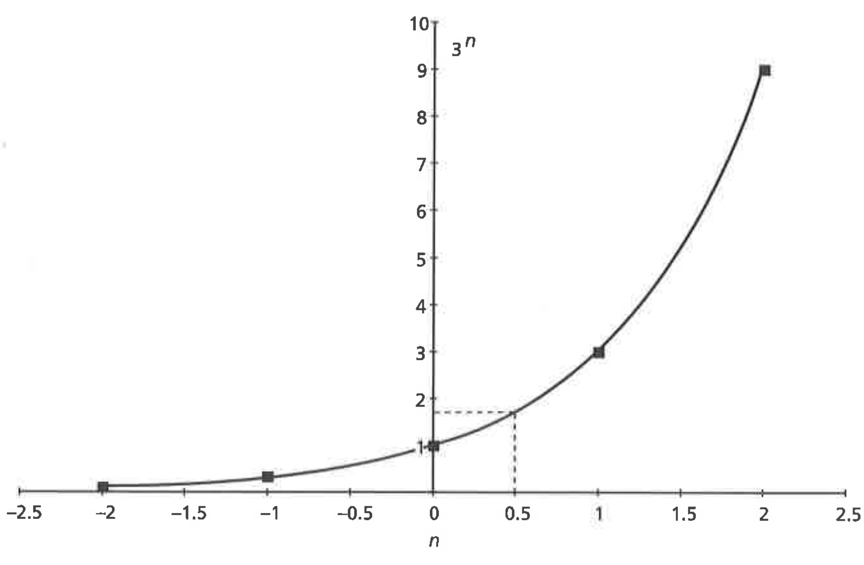


**Fig. 2.9** A smooth curve drawn through the data from Table 2.4. The dashed lines indicate the point where $n = 0.5$ from which it can be seen that $3^{1/2} \approx 1.7$.

acceptable. In Fig. 2.9 the points corresponding to $n = -2, -1, 0, 1$ and 2 are plotted as a simple graph which shows that they lie along a smooth curve. This curve can readily be used at points other than $n = -2, -1, 0, 1$ or 2. For example, at $n = 0.5$ the value on the vertical axis is about 1.7. Similarly, at $n = 1.5$ the vertical axis reads approximately 5.2. Thus $3^{0.5}$ is approximately 1.7 and $3^{1.5}$ is about 5.2. A more mathematically formal treatment of this subject is beyond the scope of this book but the main point to learn here is that it is not necessary to use integers when raising a number to a power. Negative fractional exponents are also possible. From Fig. 2.9 it can be seen that $3^{-0.5}$ is around 0.6.

Fractional powers behave in exactly the same way as integer powers. Thus, they obey the equations given in Chapter 1 for manipulating powers, i.e.

$$x^a x^b = x^{a+b} \tag{1.4}$$

$$x^a / x^b = x^{a-b} \tag{1.5}$$

and

$$(x^a)^b = x^{ab} \tag{1.6}$$

For example, from Eqn. 1.4, $x^{0.3} \times x^{0.4} = x^{0.7}$

A direct result of this is that some of these fractional powers have a very simple interpretation. For example, a number raised to the power of 0.5 is the square root of the number ($x^{1/2} = \sqrt{x}$) since $x^{1/2} \times x^{1/2} = x^1$. Similarly, a number raised to the power of one-third results in the cube root ($x^{1/3} = \sqrt[3]{x}$). Figure 2.9 indicated that $3^{1/2}$ is around 1.7; in fact, the square root of 3 is 1.732.

**Question 2.7** Draw up a table of $5^n$ for $n = -2, -1, 0, 1$ and 2. Plot the result. Hence, estimate $1/\sqrt{5}$. In fact, this can be done in two ways. First, estimate it directly from the graph. Secondly, use the graph to estimate $\sqrt{5}$ and then calculate $1/\sqrt{5}$. Compare these answers to each other and to the value given by a calculator.

I will finish this section on polynomial functions and their extensions by using a simple geological example of the use of fractional powers. There are theoretical reasons for expecting water depth, $d$, in the vicinity of a mid-ocean spreading ridge to depend upon the square root of the distance, $x$, from the ridge axis according to

$$d = d_0 + a x^{1/2} \tag{2.10}$$

where $a$ is a constant which will depend upon factors such as the spreading rate and $d_0$ is the depth of the ridge axis. Figure 2.10 shows a comparison between the depths predicted by Eqn. 2.10 and the true water depths in the

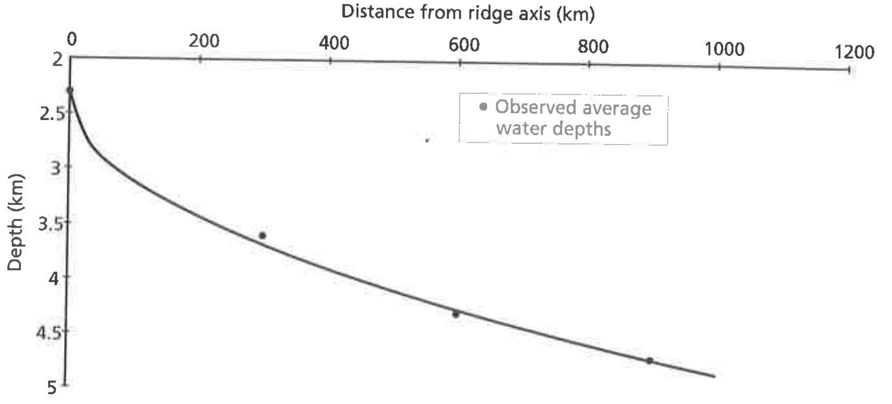


**Fig. 2.10** Ocean water depth in the vicinity of the Pacific–Antarctic Ridge. The solid line shows the predicted depth using Eqn. 2.10.

vicinity of the Pacific–Antarctic spreading ridge assuming a value for $d_0$ of 2.3 km and a value for $a$ of 0.08. Thus, at a distance from the ridge axis of 900 km, Eqn. 2.10 predicts a depth of

$$d = 2.3 + (0.08 \times \sqrt{900}) = 2.3 + (0.08 \times 30) = 2.3 + 2.4 = 4.7 \text{ km}$$

As you can see on Fig. 2.10, the true water depth is indeed very close to this value.

## 2.7 Allometric growth and exponential functions

Polynomial functions (Section 2.4) and their extensions (Sections 2.5 and 2.6) are extremely versatile and can be used to describe many situations. However, there are situations in which they are not appropriate. The way in which sediments compact as they are buried is a good example.

Water contained in recently deposited sediments is usually squeezed out as the sediments are buried. Thus, sediments start with a relatively large porosity and lose this during burial. A particularly simple approximation for the way in which this happens is to assume that a certain proportion of the water is expelled for a given amount of burial. In a particular case, half of the water might be released when the sediment is buried by 1 km and half of the remaining liquid removed during further burial to 2 km. If the sediment started with a porosity of 0.6 when deposited, the resulting porosity at various depths would be as shown in Table 2.5.

The important point about this example is that porosity always decreases with increasing burial but never actually reaches zero. It would be very difficult to reproduce this using polynomial functions. However, the values in Table 2.5 could be produced by using

| Depth (km) | Porosity |
|---|---|
| 0 | 0.6 |
| 1 | 0.3 |
| 2 | 0.15 |
| 3 | 0.075 |
| 4 | 0.0 375 |

**Table 2.5** Variation in porosity with depth assuming an initial porosity of 0.6 which halves for every kilometre of burial.

$$\phi = 0.6 \times 2^{-z} \tag{2.11}$$

in which $\phi$ is the porosity at a depth $z$ (note that porosity is nearly always denoted by the Greek letter $\phi$). For $z = 3$ km, $2^{-z}$ will be $2^{-3} = 1/8$ and therefore $\phi$ becomes $0.6 \times 1/8 = 0.075$ as shown in Table 2.5.

> **Question 2.8** What porosity does Eqn. 2.11 give at a depth of 2 km?

Now, whilst Eqn. 2.11 is similar to those discussed in Section 2.6, the crucial difference is that the variable, $z$, appears as the exponent in this expression, i.e. the power used varies as $z$ varies. Compare this to Eqn. 2.10 in which the variable, $x$, is raised to the fixed power 0.5. This subtle difference produces a rather different type of function. Its general form could be expressed as

$$y = ab^{cx} \tag{2.12}$$

where $y$ is a function of $x$ whilst $a$, $b$ and $c$ are constants. Equations such as this are called either **allometric growth laws** or **exponential functions**. This equation does not actually need three separate constants since $b^c$ is itself just another constant ($b^c = d$ say) which means that only two independent constants are needed for the general form of this equation. There are two ways of achieving this. First, simply use $b^c = d$ and write Eqn. 2.12 in the form

$$y = ad^x \tag{2.13}$$

which has two constants $a$ and $d$. Alternatively, the constant $b$ in Eqn. 2.12 is fixed to be a particular, convenient, value and $c$ is retained as an independent constant. For example, $b = 10$ may be simple to use in some contexts leading to expressions like

$$y = a \times 10^{cx} \tag{2.14}$$

The choice of value to use for $b$ is entirely arbitrary and can be varied to suit the problem. However, 99% of the time a rather peculiar choice for $b$ is made. Normally the value $b \approx 2.718$ is used! This number, denoted by the letter $e$, is special for reasons which will be touched upon in Chapter 8. For now, it is sufficient to know that it is a very important number in mathematics. It is similar to the number $\pi$ in that it is **irrational** (i.e. it cannot be expressed exactly as a fraction and, in decimal form, the number goes on for ever); it crops up in many different branches of mathematics and the use of $e$ to denote this number is sufficiently universal that it will not, normally, be defined in most papers or books. Thus, another important form for Eqn. 2.12 is

$$y = a\, e^{cx} \tag{2.15}$$

in which $e \approx 2.718$. An alternative way of writing this is

$$y = a \exp(cx) \tag{2.16}$$

which means exactly the same thing as Eqn. 2.15. (N.B. 'exp' here is a single word, it does not mean $e$ times $p$. In fact, 'exp' is an abbreviation for exponential.)

Spreadsheet *Exp.xls* allows you to plot Eqn. 2.12 for any value of $a$, $b$ or $c$. In particular, if $c$ is set to 1 then it models Eqn. 2.13 and, if $b$ is set to 10 then it will plot Eqn. 2.14 for you. To get a standard exponential function (i.e. Eqn. 2.15) you should set $b$ to 2.718 or, alternatively, type the formula = exp(1) into cell B9.

Equation 2.15 is frequently the form used for modelling the variation in porosity with depth. This leads to expressions such as

$$\phi = \phi_0\, e^{-z/\lambda} \tag{2.17}$$

An example is the best way to illustrate this. If the constants have the values $\phi_0 = 0.7$, $\lambda = 2$ km, the porosity at a depth of 4 km would be

$$\begin{aligned}\phi &= 0.7 \exp(-4/2)\\ &= 0.7 \exp(-2)\\ &= 0.7 \times 0.135\\ &= 0.0945\end{aligned}$$

**Question 2.9** What porosity would this predict for $z = 1$ km?

It is worth spending a little time considering the meaning of the constants $\phi_0$ and $\lambda$. These do have fairly simple interpretations. First, remember that any number to the power zero equals one. Thus, if the depth is zero, Eqn. 2.17 becomes

$$\begin{aligned}\phi &= \phi_0 \exp(-0/\lambda)\\ &= \phi_0 \exp(0)\\ &= \phi_0 \times 1.0\\ &= \phi_0\end{aligned}$$

In other words, $\phi_0$ is simply the porosity at zero depth. The meaning of $\lambda$ can be seen by setting $z$ to be $\lambda$ kilometres. Equation 2.17 then gives

$$\begin{aligned}\phi &= \phi_0 \exp(-\lambda/\lambda)\\ &= \phi_0 \exp(-1)\\ &= \phi_0/e\\ &= \phi_0/2.71\end{aligned}$$

i.e. $\lambda$ is the depth at which the porosity reduces to around one-third of its starting value.

## 2.8 Logarithms

The logarithmic function is the final type of relationship which will be investigated in this chapter. Logarithms solve the problem of how to rearrange equations of the form $y = a^x$ (i.e. exponential or allometric growth functions, Section 2.7) into an equation for $x$ in terms of $y$. The solution is $x = \log_a(y)$. In other words, logarithms are defined as the **inverse** of exponential functions. For example, if $y = 10^3 = 1000$, then $\log_{10}(1000) = 3$. Similarly, $\log_{10}(100\,000) = 5$ since $100\,000 = 10^5$. Tables 2.6 and 2.7 make the same point in a slightly different way. Table 2.6 lists the result of raising 10 to the power of various integers, i.e. $10^n = 100$ if $n = 2$ etc. The definition given for logarithms above, implies that Table 2.6 could be rewritten as a table of logarithms simply by swapping around the columns (i.e. Table 2.7).

| $n$ | $10^n$ |
|---|---|
| –2 | 0.01 |
| –1 | 0.1 |
| 0 | 1 |
| 1 | 10 |
| 2 | 100 |
| 3 | 1000 |

**Table 2.6** Ten raised the power of the integers between –2 and 3.

| Number | Logarithm |
|---|---|
| 0.01 | –2 |
| 0.1 | –1 |
| 1 | 0 |
| 10 | 1 |
| 100 | 2 |
| 1000 | 3 |

**Table 2.7** Table of logarithms produced by swapping the columns in Table 2.6.

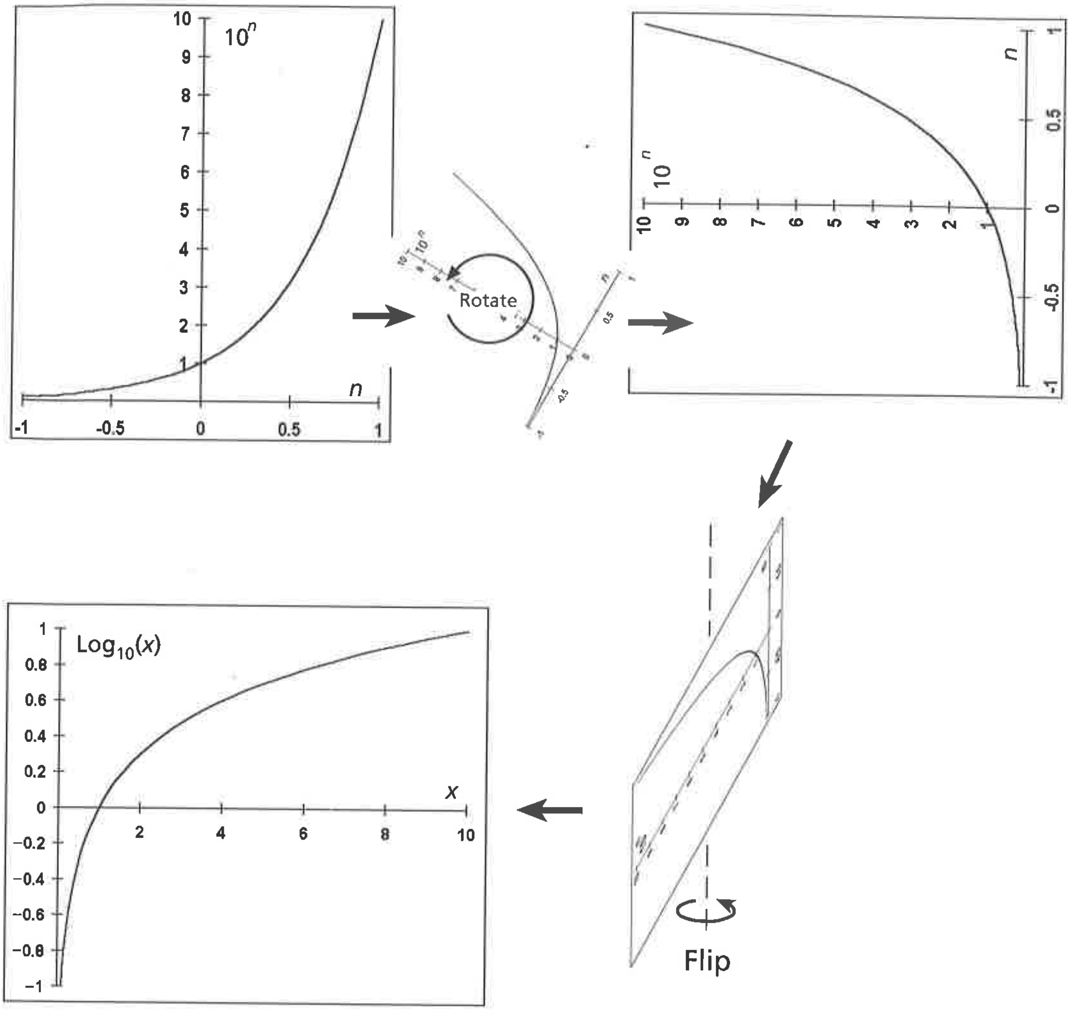


Fig. 2.11 The graph at top left shows the curve $y = 10^n$ using the data from Table 2.6. The graph at bottom left shows the curve $y = \log(x)$ using the data from Table 2.7. This figure shows that the two curves are exactly the same shape since the two functions are very closely related.

Figure 2.11 makes the same point in yet another way. The top left-hand graph shows a graph of $10^n$ versus $n$ whilst the graph at lower left is of $\log_{10}(x)$ versus $x$. However, as Fig. 2.11 attempts to make clear, these two functions are very closely related since the two graphs show the same curve just plotted in a different orientation.

Another important point about Fig. 2.11 is that the logarithmic curve never crosses the vertical axis. If this curve was plotted for smaller values of $x$ it would get even steeper and it would never reach the $\log_{10}(x)$ axis. As a consequence, logarithms of negative numbers do not exist. If you try to use a calculator, for example to find $\log_{10}(-2)$, you will get an error message.

| Fault length (m) | Number |
|---|---|
| 0.001 | 10 109 |
| 0.01 | 957 |
| 0.1 | 132 |
| 1 | 11 |
| 10 | 1 |

**Table 2.8** Number of faults of length greater than a given size at a particular outcrop. For example there are 11 faults of length 1 m or longer.

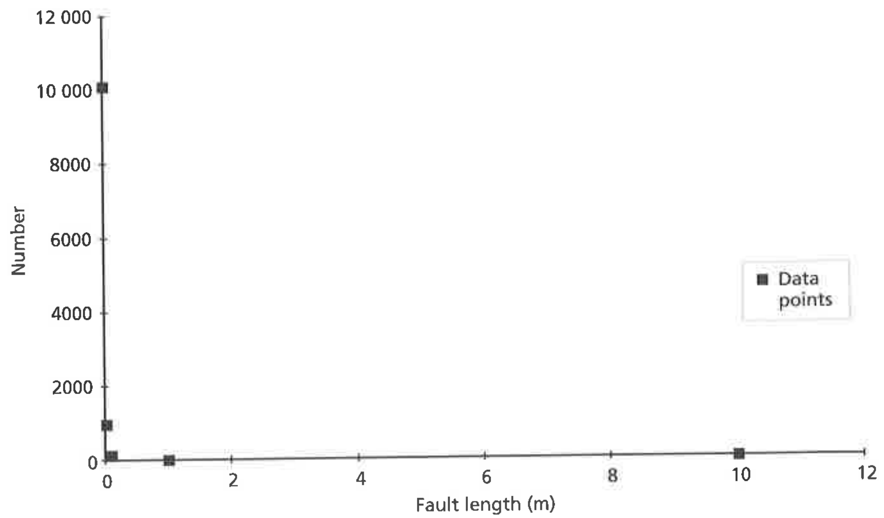


**Fig. 2.12** A simple graph of the data from Table 2.8.

There are three main uses for logarithms.

**1** Rearranging equations containing exponential functions as discussed above.

**2** Reducing exponential and allometric functions to simple straight lines.

**3** Compressing large data ranges.

The first two of these are closely related and will be covered in more detail in the next chapter. What about the third use, i.e. compressing large ranges? Fault sizes provide a good example. Faults occur on a vast range of scales from millimetres long to hundreds of kilometres long. Now, in a particular area, the number of faults of different sizes might be something like Table 2.8 in which the fault length is tabulated against the number of faults observed of this size or larger. Note that such tables typically show that small faults are much more common than larger faults. If we attempt to plot these data on a graph, the result is as shown in Fig. 2.12.

This graph is not very helpful because all the points lie on or near the axes. The problem is the large range of values that occurs; some values are very

**Table 2.9** Result of taking logarithms of the data in Table 2.8.

| Fault length (m) | log(length) | Number | log(number) |
|---|---|---|---|
| 0.001 | −3 | 10 109 | 4.00 |
| 0.01 | −2 | 957 | 2.98 |
| 0.1 | −1 | 132 | 2.12 |
| 1 | 0 | 11 | 1.04 |
| 10 | 1 | 1 | 0.00 |

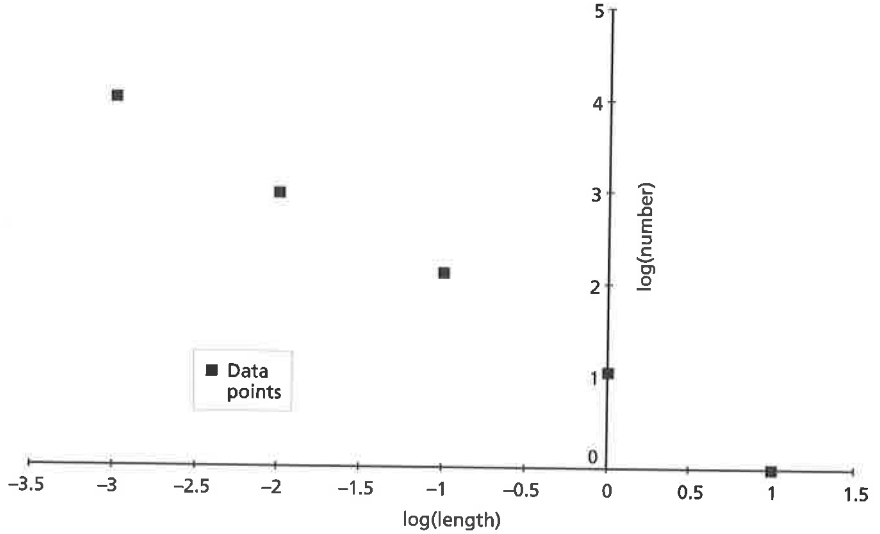


**Fig. 2.13** Graph of the logarithm data from Table 2.9.

small whilst some are very large. This makes it impossible to find axes scales which enable all the data to be properly viewed. However, if we add logarithms to the columns in Table 2.8 (to give Table 2.9) and plot these instead, Fig. 2.13 results. Clearly, this graph is much more informative since the data are now spread more evenly across it.

Thus, in summary, a major use for logarithms is for plotting graphs of quantities which vary over large ranges. This is something which occurs very frequently in geological problems.

## 2.9 Logarithms to other bases

The previous section explained how logarithms were obtained from a table showing the number 10 raised to various powers. However, it is not necessary for the number 10 to be used. If other numbers are used then the result is a logarithm in a different base. An alternative choice might be, for example, a

| $n$ | $6^n$ |
|---|---|
| −2 | $1/6^2 = 1/36 = 0.0278$ |
| −1 | $1/6^1 = 1/6 = 0.167$ |
| 0 | $6^0 = 1$ |
| 1 | $6^1 = 6$ |
| 2 | $6^2 = 36$ |

Hence:

| $x$ | $\log_6(x)$ |
|---|---|
| 0.0278 | −2 |
| 0.167 | −1 |
| 1 | 0 |
| 6 | 1 |
| 36 | 2 |

**Table 2.10** The result of raising 6 to the power of various integers and the resulting table of logarithms obtained by swapping the columns around.

base of 6. Table 2.10 shows the number 6 raised to the power of various integers and the resulting table of logarithms obtained by swapping the columns around.

Note that, in order to indicate that these are logarithms to base 6, a 6 subscript is written after the word 'log'. Logarithms to base 10, i.e. those discussed in section 2.8, are frequently written without this subscript so that, if the subscript is missing, logs to base 10 should be understood. Logarithms to base 10 are sometimes called **common logarithms**.

> **Question 2.10** What number has a logarithm, in base 5, of 2 (i.e. if $\log_5(x) = 2$, what is $x$)? Hint: Construct a table similar to Table 2.10 but for a base of 5.

A commonly used scale for quantifying sediment grain sizes is called the phi scale and this uses logarithms to base 2. The formal definition of the phi grain size is

$$\phi = -\log_2(d) \tag{2.18}$$

where $d$ is the grain size in millimetres. This is not as complex as it sounds as a table of base 2 logarithms shows (Table 2.11).

To convert phi values into grain sizes in millimetres it is only necessary to start at 1 mm and halve this $\phi$ times (e.g. for $\phi = 3$, halving 1.0 mm three times gives a grain size of 1/8 mm). A convention that halving a negative number of times means doubling the same number of times must also be used (e.g. for $\phi = -3$, doubling 1.0 mm three times gives a grain size of 8 mm).

**Table 2.11** Logarithms to base 2. Note that the log increases by one for each doubling of $x$ and $\log_2(1) = 0$.

| $x$ | $\log_2(x)$ |
|---|---|
| 0.25 | −2 |
| 0.5 | −1 |
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 8 | 3 |

However, for grain sizes which are not an integer power of 2 (0.25, 0.5, 1, 2, 4, 8 mm, etc., are all integer powers of 2) this procedure will not work and the formula given by Eqn. 2.18 must be used. Many calculators allow you to do this directly but, if you do not have access to such a calculator, there is a simple recipe for converting between logarithm bases:

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)} \tag{2.19}$$

Converting a logarithm to the base 2 into a common logarithm should make this clearer. If $b = 2$ and $c = 10$, Eqn. 2.19 becomes

$$\begin{aligned}\log_2(a) &= \frac{\log_{10}(a)}{\log_{10}(2)} = \log_{10}(a)/0.301 \\ &= 3.32\ \log_{10}(a)\end{aligned} \tag{2.20}$$

i.e. use the common logarithm and then multiply by 3.32. So, a grain size of 2.3 mm would have a phi value given by

$$\begin{aligned}\phi &= -\log_2(2.3) \\ &= -3.32\ \log_{10}(2.3) \\ &= -3.32 \times 0.362 \\ &= -1.20.\end{aligned}$$

To finish this chapter I'll discuss a particularly common base for logarithms, namely logarithms to base $e$ (remember $e \approx 2.718$). These are known as natural logarithms and are denoted either by using the subscript $e$ (natural log of $x$ is written $\log_e(x)$) or, more commonly, it is denoted by ln (i.e. natural log of $x$ is written $\ln(x)$). This type of logarithm probably occurs more frequently than any other, and will be used throughout this book, so it is important to be familiar with its appearance.

Spreadsheet *Log.xls* allows you to plot the common logarithm function, the natural logarithm function and a logarithm function assuming a user-defined base. Using this, for example, you can investigate the $\log_2$ function.

## 2.10 Further questions

**2.11** The following data were taken from the Troll 3.1 well in the Norwegian North Sea.

| Depth (cm) | Age (years) |
|---|---|
| 19.75 | 1 490 |
| 407.0 | 10 510 |
| 545.0 | 11 160 |
| 825.0 | 11 730 |
| 1158.0 | 12 410 |
| 1454.0 | 12 585 |
| 2060.0 | 13 445 |
| 2263.0 | 14 685 |

By plotting a graph of these data, estimate:
(i) the sedimentation rate for the last 10 000 years;
(ii) the sedimentation rate for the preceding 5000 years;
(iii) the time since sedimentation ceased.
(Data taken from Lehman, S. and Keigwin, L. (1992) Sudden changes in North Atlantic circulation during the last deglaciation. *Nature*, 356, 757–62.)

**2.12** As crystals settle out of magmas, element concentrations, $C$, in the remaining liquid change according to the equation

$$C = C_0F^{(D-1)}$$

where $C_0$ is the concentration of the element in the liquid before crystallization began, $F$ is the fraction of liquid remaining and $D$ is a constant (known as the distribution coefficient). Calculate the concentration of an element after 50% crystallization (i.e. $F = 0.5$) if its initial concentration was 200 ppm and $D = 6.5$.

**2.13** Radioactive minerals become less active with time according to the equation

$$\ln(a) = \ln(a_0) - \lambda t$$

where $a$ is the radioactivity, $a_0$ is the initial radioactivity, $t$ is time and $\lambda$ is a constant which depends upon the mineral. If $a_0 = 1000$ counts per second and $\lambda = 10^{-7}\ \text{y}^{-1}$, draw up a table and plot a graph of $\ln(a)$ against $t$ for times ranging from 0 to 100 My. From your graph, estimate the age of a specimen which has decayed to $a = 100$ counts per second.

**2.14** The variation in gravitational strength with altitude should obey the equation

$$g = g_0 + ah$$

where $g$ is the measured strength of gravity, $g_0$ is the gravitational strength at sea level, $a$ is a constant and $h$ is height above sea level. However, the presence of metallic ore bodies, volcanic intrusions, etc., tend to increase the local strength of gravity slightly. Thus, real gravitational measurements do not quite obey this expression. Deviations from this equation can therefore be used to indicate the presence of such features. Using the figures given below, plot a graph of $g$ against $h$ and hence estimate $g_0$ and $a$. Hence, calculate the deviation of each measurement from its expected value. Plot a graph of this deviation as a function of position and determine the approximate extent of an ore body known to outcrop in this area.

| Horizontal position, $x$ (km) | Altitude, $h$ (m) | Gravity, $g$ ($m\ s^{-2}$) |
|---|---|---|
| 0.0 | 150 | 9.80 945 |
| 0.5 | 100 | 9.8 097 |
| 1.0 | 170 | 9.80 949 |
| 1.5 | 200 | 9.8 094 |
| 2.0 | 150 | 9.80 955 |
| 2.5 | 130 | 9.80 951 |
| 3.0 | 120 | 9.80 954 |

**2.15** The rate of accumulation, $p$, of carbonate sediments on a reef is given approximately by

$$p = p_0 \exp(-z/Z)$$

where $p_0$ and $Z$ are constants and $z$ is depth below sea level.

(i) Calculate $p$ at depths of 0, 2, 4, . . . , 20 m if $p_0 = 3$ m $ky^{-1}$ and $Z = 20$ m. Sketch the results.

(ii) Give an interpretation for the constants $p_0$ and $Z$.