CHAPTER 8 Stress

We have described a variety of structures that can form in rocks as a result of brittle deformation. Knowing what structures exist, we are naturally inclined to ask why they exist. What caused them to form? What does their existence tell us about the processes operating in the Earth at the time they formed?

Our experience tells us that things break when too much force is applied to them. Thus we must consider what happens when forces are applied to a body of rock. In doing so, we are led to the concept of stress as a means of describing the physical state of material to which forces are applied.

8.1 Preview

The concept of stress can initially be confusiug, parrly because quantities that require several numbers to represent them are unfamiliar, and partly because the notation that we must use to represent these quantities is unfamiliar. In fact, however, the physical idea of stress is not difficult. Using two-dimensional geometry, we briefly introduce the physical ideas leading to the concept of stress (see Table 8.1).

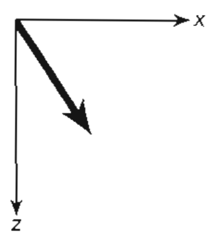
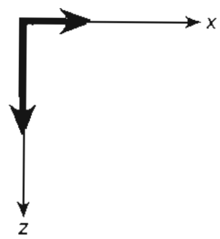
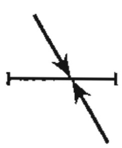
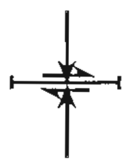
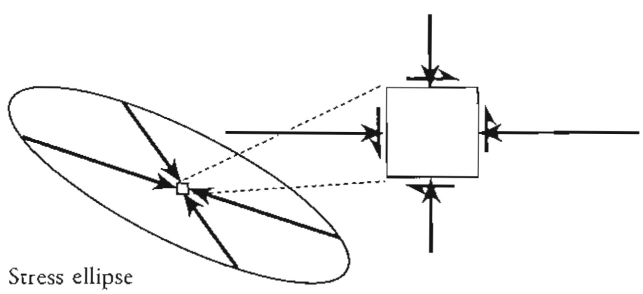
We start with the idea of force because it is the basic concept and because we all have a physical intuition of what force is from our everyday experience of pushing and pulling on things. Force is a vector quantity which has a magnitude (how hard the push is) and direction (which way the push is), and it is diagrammed by an arrow (*A* in Table 8.1). In a given system of coordinates (x, z) a force vector can be represented by components parallel to each of the coordinate axes (see Box 8.1).

The intensity of the force depends on the area of the surface over which the force is distributed. It is called a traction, and it has units of force per unit area (*B* in Table 8.1). The larger the area over which a given force is distributed, the smaller the traction on that surface. Thus the weight of a gallon of water produces a higher traction on the bottom of a tank 0.5 m on a side than on the bottom of a tank 1 m on a side. The traction is commonly represented in terms of its components perpendicular and parallel to the surface on which it acts.

In order to satisfy the requirements of equilibrium, any surface must have a pair of equal and opposite tractions acting on opposite sides of the surface. This pair of tractions defines the surface stress, which is commonly represented by a pair of equal and opposite components acting perpendicular to the surface and another pair acting parallel to the surface (*C* in Table 8.1).

For a given system of forces applied to a body of material, the surface stress at a given point varies with the orientation of the surface through the point. In order to know the effect at a point of all the forces acting on the body, we must be able to determine the surface stress on any plane through the point. Imagine,

Table 8.1 Development of the Concept of Stress

Diagrams		Definitions
A. Force 	Force components 	A push or a pull
B. Traction 	Traction components 	Force per unit area on a surface of a specified orientation (a measure of force intensity)
C. Surface stress 	Surface stress components 	A pair of equal and opposite tractions acting across a surface of specified orientation
D. Stress at a point (two-dimensional)  Stress ellipse	Stress tensor components (two-dimensional)	Surface stresses on planes of all orientations through a point (represented by the stress ellipse) Two of the surface stresses acting on two perpendicular surfaces, respectively, suffice to define the entire ellipse.

example, a cube that is compressed perpendicular to its faces by three vices each applying a different force. The surface stress on each pair of cube faces would be different, and each is independent of the surface stresses on the other two pairs of faces. It turns out that the surface stress on any other orientation of plane through the center of the cube can be determined from these three independent surface stresses. In fact, if we know the surface stresses on any three mutually perpendicular planes through a point, we can calculate the surface stress on any other plane through that point. The components of these three surface stresses measnred perpendicular and parallel to their respective planes make up the components of the stress tensor. Thus the stress tensor is a quantity that simply permits us to calculate the surface stress on a plane of any possible orientation at a given point. If we know that, we know completely what the material "feels" at that point as a result of the forces applied to the body.

In two dimensions, if we plot from a common origin the surface stresses for all the orientations of surfaces at a point, they define an ellipse (*D* in Table 8.1), which is therefore a complete representation of the two-dimensional stress tensor. The size, shape, and orientation of the ellipse are completely defined if we know the surface stresses on any pair of perpendicular planes through the point. The components of these two surface stresses are the components of the two-dimensional stress tensor (*D* in Table 8.1).

Fundamentally, that is all there is to the idea of stress. In this chapter, we further develop these concepts and the notation to express them.

8.2 Force, Traction, and Stress

Force

Because we are concerned with vectors such as force, we review in Box 8.1 some basic properties of vectors. Forces applied to a body and originating outside the body are of two types:

1. Body forces act on each particle of mass, independent of the surrounding material. By far the most important body force to a structural geologist is the Earth's force of gravity. It exerts on each volume of rock a force that is proportional to the mass within that volume.
2. Surface forces arise either from the action of one body on another across the surface of contact between them or from the action of one part of a body on another part across an internal surface. For example, if our hand pushes on the end of a block of rock, we apply a surface force across the area of contact between our hand and the block. Moreover, across any internal surface of arbitrary orientation that divides the block in two, one side of the block applies a surface force on the other side.

For the present discussion we focus our attention on surface forces.

The Traction: A Measure of Force Intensity

A large force clearly has a greater effect on an object than a small force. But just knowing how much force is applied to a body does not give us all the information we need to determine how a deformable body will respond. For example, a thick wooden pillar might easily support the force exerted by a large mass (Figure 8.1*A*); that same force, however, would break the thin wooden leg of a table (Figure 8.1*B*). Because the type of material supporting the force is the same in both cases, we expect that the *intensity* of the force must be higher on the table leg than on the pillar, even though the magnitude of the force is the same.

The traction $\mathbf{\Sigma}$ is the force intensity, and it is defined by dividing the force applied $\mathbf{F}$, by the area A across which it acts.[1] It therefore has physical units of force per unit area.[2] Figure 8.2*A* shows one force $\mathbf{F}^{(top)}$ acting on the top side of the surface whose area is A, and another $\mathbf{F}^{(bottom)}$ acting on the bottom side. Figure 8.2*B* shows the corresponding tractions $\mathbf{\Sigma}^{(top)}$ and $\mathbf{\Sigma}^{(bottom)}$ that act on the opposite sides of the surface. In some sources, the traction is called the stress vector. We avoid this usage because the quantity is not a true vector, as we see below, and because using the same word for both traction and stress blurs the distinction between them.

[1] We generally represent scalars like the area A in italic type; vectors like the force $\mathbf{F}$, and tensors like the stress $\boldsymbol{\sigma}$, as well as tractions and surface stresses, in boldface type.

[2] The units of the traction are the same as for the hydrostatic pressure on a surface. The two quantities differ in that hydrostatic pressure is always perpendicular to the surface on which it acts, whereas the traction in general is not.

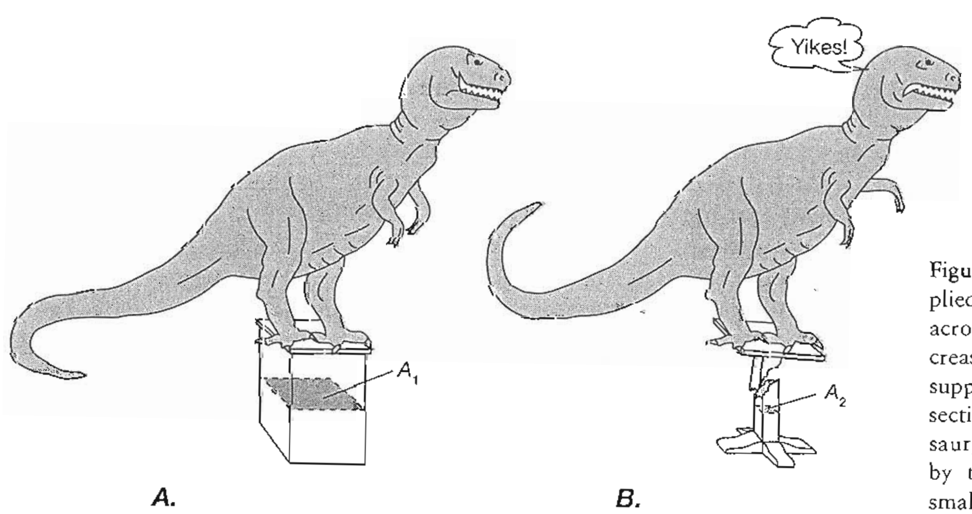


Figure 8.1 The intensity of an applied force increases as the area across which it is distributed decreases. *A*. A tyrannosaur is happily supported on a large pillar of cross-sectional area A_1. *B*. The tyrannosaur, to her dismay, is not supported by the table leg having a much smaller cross-sectional area A_2.

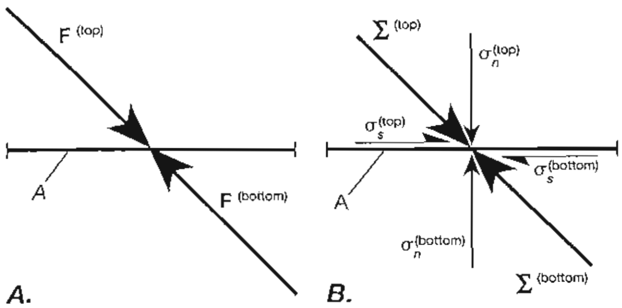


Figure 8.2 Force, traction, and surface stress. *A.* A force $F^{(top)}$ applied to the top of the surface of area *A* is balanced by an equal and opposite force $F^{(bottom)}$ on the bottom of the surface. *B.* The force intensity is given by the associated tractions $\Sigma^{(top)}$ and $\Sigma^{(bottom)}$ which are equal and opposite. Each traction can be expressed in terms of its normal component σ_n and its shear component σ_s. The balanced pair of tractions is the surface stress; the balanced pairs of components are the normal stress component and the shear stress component.

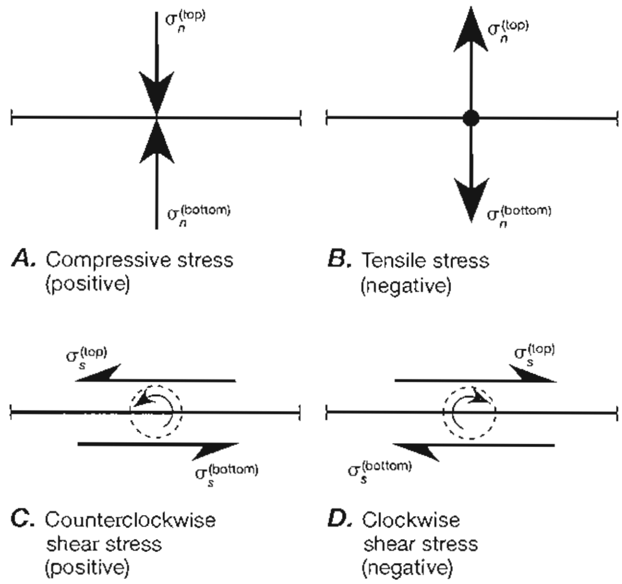


Figure 8.3 The Mohr circle sign conventions for the components of the stress at a point.

It is usually convenient to resolve the traction into two components, one perpendicular to the surface on which it acts and the other parallel to that surface. These components are, respectively, the normal traction component σ_n and the shear traction component σ_s (Figure 8.2*B*).

If the force is uniformly distributed over a large area *A*, and F represents the total force, then the traction on the whole area is given by

$$\Sigma \equiv \frac{F}{A} \qquad (8.1)$$

If the force is nonuniformly distributed over the area, that is, if it changes direction and magnitude across the surface, then we can define the traction only *at a point* on the surface. We represent the point as an infinitesimal area d*A* of the surface on which an infinitesimal part of the total force dF acts.

$$\Sigma \equiv \frac{dF}{dA} \qquad (8.2)$$

The magnitude and direction of the traction can then vary from point to point across the surface.

The Surface Stress

We require the surface to be in mechanical equilibrium, which means it cannot accelerate independently of the material in which it lies. For this to be true, according to Newton's second law, opposing forces must exist on opposite sides of the surface such that all the forces on the surface sum to zero (Figure 8.2*A*):

$$F^{(top)} + F^{(bottom)} = 0$$

We can express this balance of forces in terms of the tractions by dividing the forces by the area across which they act.

$$\frac{F^{(top)}}{A} + \frac{F^{(bottom)}}{A} = 0$$

$$\Sigma^{(top)} + \Sigma^{(bottom)} = 0$$

$$\Sigma^{(top)} = -\Sigma^{(bottom)} \qquad (8.3)$$

Equation (8.3) asserts that the tractions on the top and bottom of the surface must be equal and opposite; the same relationship must therefore apply individually to the normal traction and shear traction components (Figure 8.2*B*).

$$\sigma_n^{(top)} = -\sigma_n^{(bottom)} \qquad \sigma_s^{(top)} = -\sigma_s^{(bottom)} \qquad (8.4)$$

The surface stress, or each of its components, consists of a pair of equal and opposite tractions, or a pair of equal and opposite traction components, acting on a surface. If the two equal and opposite normal traction components, $\sigma_n^{(top)}$ and $\sigma_n^{(bottom)}$, point toward each other, they define a compressive stress which tends to press the material together across the surface (Figure 8.3*A*). If they point away from each other, they define a tensile stress which tends to pull the material apart across the surface (Figure 8.3*B*). We consider that *compressive stresses are positive* and *tensile stresses are negative.*

Two equal and opposite shear traction components, $\sigma_s^{(top)}$ and $\sigma_s^{(bottom)}$, define a shear stress or a shear couple. The shear stress may be clockwise or counterclockwise, depending on which way a ball would turn if it were placed between the two arrows repre-

Box 8.1 What is a Vector? A Brief Review

A vector is a quantity that has both a magnitude and a direction. A scalar quantity, on the other hand, has only a magnitude. Temperature and mass density are familiar physical quantities that are scalars. These quantities are each represented by a single number that has no directional quality associated with it, such as 35°C and 2500 kg/m^3. Familiar examples of vector quantities include velocity and force. We can define a vector quantity completely only by giving its magnitude *and* the direction in which it acts: A plane travels 400 km/hr in a horizontal northeast direction. Vectors can be represented diagramatically by arrows. The length of the arrow shaft is made proportional to the magnitude of the vector, and the direction of the shaft and point indicates the direction of the vector.

Two vectors can be added using the parallelogram rule. If, for example, we wish to add two forces **V** and **W** that act on a point p, we draw the arrows representing the forces tail to tail and construct a parallelogram with the arrows defining two adjacent sides (Figure 8.1.1). The sum of the forces, called the resultant force **R**, is then the vector from the common origin to the diagonally opposite corner of the parallelogram. Thus the effect of applying the forces **V** and **W** to p is the same as if the resultant force **R** were applied to p.

In order to specify a direction, it is necessary to have some frame of reference, such as the geographic coordinates north, east, and down (which we used above to describe the velocity of the airplane). The frame of reference in three-dimensional space is commonly taken to be a mutually orthogonal system of coordinates, and we assume that its orientation is known. We label the axes x_1, x_2, and x_3 according to the *right-hand rule*. By this rule, if the fingers of the right hand are oriented to curve along the direction of rotation from positive x_1 to positive x_2, then the thumb points along positive x_3 (Figure 8.1.2). These axes are also often labeled x, y, and z, but it is more convenient to use the subscript numbers.

If we consider a vector **V** to represent, for example, a force in three-dimensional space (Figure 8.1.3), using the parallelogram rule shows that it can be considered the resultant of two forces: one, $\mathbf{V}_3$, parallel to the x_3 axis and the other, **W**, lying in the x_1–x_2 plane.

$$\mathbf{V} = \mathbf{W} + \mathbf{V}_3 \qquad (8.1.1)$$

Using the parallelogram rule again for **W** shows that it can be considered the resultant of two forces $\mathbf{V}_1$ and $\mathbf{V}_2$, which parallel the x_1 and x_2 axes, respectively.

$$\mathbf{W} = \mathbf{V}_1 + \mathbf{V}_2 \qquad (8.1.2)$$

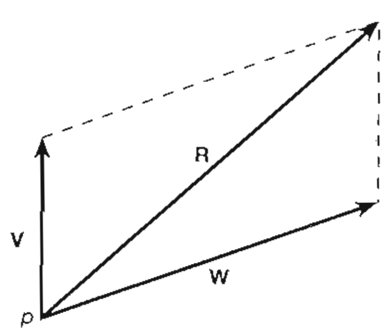


Figure 8.1.1 The parallelogram rule for vector addition.

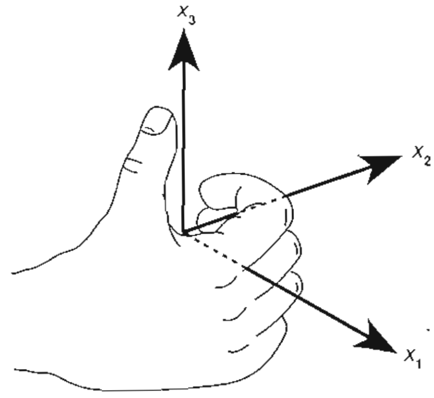


Figure 8.1.2 The right-hand rule defining the orientation of axes in a right-handed orthogonal Cartesian coordinate system.

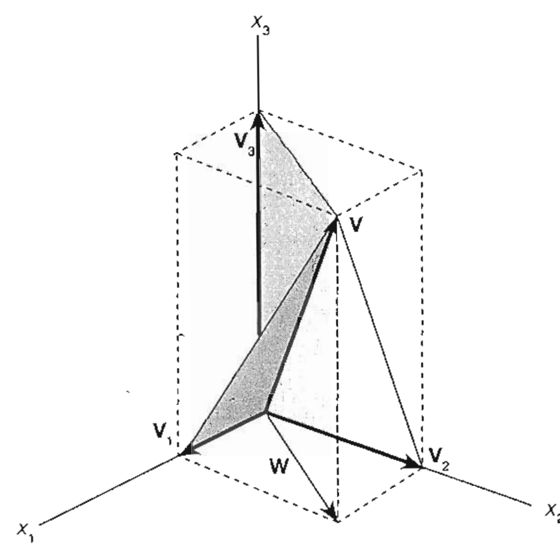


Figure 8.1.3 A vector V and its vector components (V_1, V_2, V_3) in three-dimensional space. W is the projection of V on the x_1–x_2 plane.

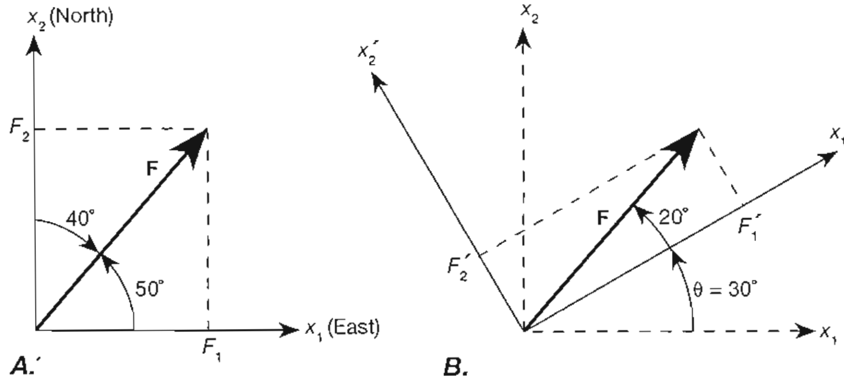


Figure 8.1.4 The dependence of the scalar components of the vector on the orientation of the coordinate system. *A.* Components (F_1, F_2) of F in the x_1–x_2 coordinate system. *B.* Components (F_1', F_2') of F in the x_1'–x_2' coordinate system.

Combining Equations (8.1.1) and (8.1.2) shows that the force **V** is the resultant of three forces, each acting parallel to one of the coordinate axes.

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \qquad (8.1.3)$$

$\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are the component vectors of **V**. If we designate their lengths by V_1, V_2, and V_3, respectively, then these are called the scalar components, or just the components, of the vector **V** in the given coordinate system. By convention, the components are always written in order. Thus the vector **V** can be represented by an ordered array of three scalar components (V_1, V_2, V_3).

For a fixed vectorial quantity such as a given force, the values of the components representing that vector quantity depend not only on the magnitude and direction of the quantity but also on the orientation of the coordinate system in which the components are defined.

The problem is simpler to explain in two dimensions for which the reference coordinates are x_1 positive due east and x_2 positive due north (Figure 8.1.4*A*). If **F** is a force of 100 N (newtons) acting 40° east of north (or, equivalently, 50° north of east), then the force vector is completely defined by its components (F_1, F_2) in the x_1–x_2 coordinate system:

$$(F_1, F_2) = (64.3, 76.6)\text{ N}$$

where

$$F_1 = |\mathbf{F}| \cos 50° = (100)(0.643) = 64.3\text{ N}$$
$$F_2 = |\mathbf{F}| \sin 50° = (100)(0.766) = 76.6\text{ N} \qquad (8.1.4)$$

If, however, we use a coordinate system x_1'–x_2', where x_1' is 30° counterclockwise from x_1 (Figure 8.1.4*B*), then exactly the same force vector **F** has components given by

$$(F_1', F_2') = (94.0, 34.2)\text{ N}$$

where

$$F_1' = |\mathbf{F}| \cos 20° = (100)(0.940) = 94.0\text{ N}$$
$$F_2' = |\mathbf{F}| \sin 20° = (100)(0.342) = 34.2\text{ N} \qquad (8.1.5)$$

For a given vector **F**, the components in different coordinate systems are systematically related. If, in Figure 8.1.5, we designate the angle between x_1 and x_1' and the angle between x_2 and x_2' by θ, then using the sides of the shaded triangles, it is not difficult to show that

$$F_1' = F_1 \cos\theta + F_2 \sin\theta$$
$$F_2' = -F_1 \sin\theta + F_2 \cos\theta \qquad (8.1.6)$$

The same situation exists in the more general three-dimensional case. Although the equations are slightly more complicated, the principle is the same: The vector **F** is the physical quantity, such as force, and it is represented by a different ordered set of components in each different coordinate system.

Because Equations (8.1.6) enable us to transform the component values from one known coordinate system to another, they are called the transformation equations. For a quantity to be a vector, its components must transform according to the rule given in these equations for two dimensions or in comparable equations for three dimensions.

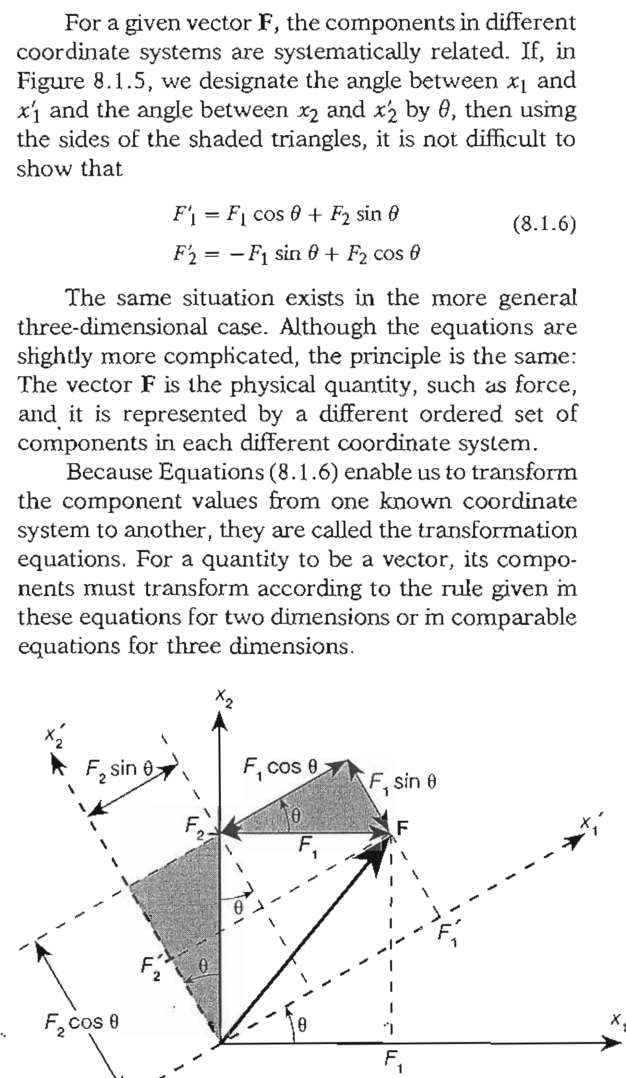


Figure 8.1.5 Geometric relationships between the scalar components of the same vector in two differently oriented coordinate systems. The sides of the shaded triangles can be used to deduce the values of the components (F_1', F_2') from the components (F_1, F_2) and the angle θ.

senting the shear stress components (Figure 8.3*C*, *D*). We consider that *counterclockwise shear couples are positive* and *clockwise shear couples are negative*.[3]

Generally, we use the symbol Σ to refer to both the traction and the surface stress, and we use the symbols σ_n and σ_s to refer to the components for both quantities. It is important to realize, however, that a surface stress is defined by *pairs* of equal and opposite traction components acting across a surface. The absolute values of the traction components and of the associated surface stress components are the same; the two differ, however, in the sign convention for the components. For example, a compressive surface stress is positive, but it is defined by one positive and one negative traction component. We will normally deal with stress components, unless it is important to consider one particular traction of the pair that defines the stress. In that case, we will identify the traction by using a superscript, such as (top) and (bottom) in Equation (8.4).

[3] We refer to this sign convention as the Mohr circle sign convention, because it is used for plotting stress components as a Mohr circle, which we discuss in Section 8.3. The sign convention is not unique, however, because the same shear stress looks clockwise and counterclockwise when viewed from opposite directions. For this reason, it differs from the sign convention used for the components of the stress tensor. We discuss the tensor sign convention in Section 8.4, and the origin of the vexing but unavoidable difference between the Mohr circle and the tensor sign conventions in Section 8.5.

A Numerical Example

As an illustration, let us calculate the components of the surface stress acting on two different planes in the pillar supporting the tyrannosaur in Figure 8.1*A*. First we determine the surface stress components on a horizontal cross section of the pillar (Figures 8.1*A*, 8.4*A*). Suppose the area of the pillar cross section is $A = L \times L = 2\ \text{m}^2$ and the tyrannosaur weighs $W = 80{,}000$ N, which is the magnitude of a force acting downward. The force per unit area that the upper part of the pillar exerts on the lower part, across the area *A*, is the traction, and the lower part exerts an equal and opposite traction on the upper part. The magnitude of the surface stress is just the same as the magnitude of the traction[4] (Figure 8.4*B*). We choose the signs of

[4] A newton is the amount of force required to accelerate 1 kilogram of mass at 1 meter per second per second ($1\ \text{N} = 1\ \text{kg m/s}^2$). Appropriately enough, a force of 1 newton is approximately equal to the weight of an apple (1 N = 0.225 lb). Newtons per square meter (N/m^2), pascals (Pa), megapascals (MPa), bars (b), and kilobars (kb) are all units of stress—that is, force per unit area—related by $10^6\ \text{N/m}^2 = 10^6\ \text{Pa} = 1\ \text{MPa} = 10\ \text{b} = 0.01\ \text{kb}$. The pressure of 1 Pa is approximately the pressure created by grinding one apple into applesauce and spreading it in an even layer over an area of $1\ \text{m}^2$. It is a rather small pressure. Atmospheric pressure ($14.7\ \text{lb/in}^2$) is approximately 10^5 Pa, 0.1 MPa, or 1 b.

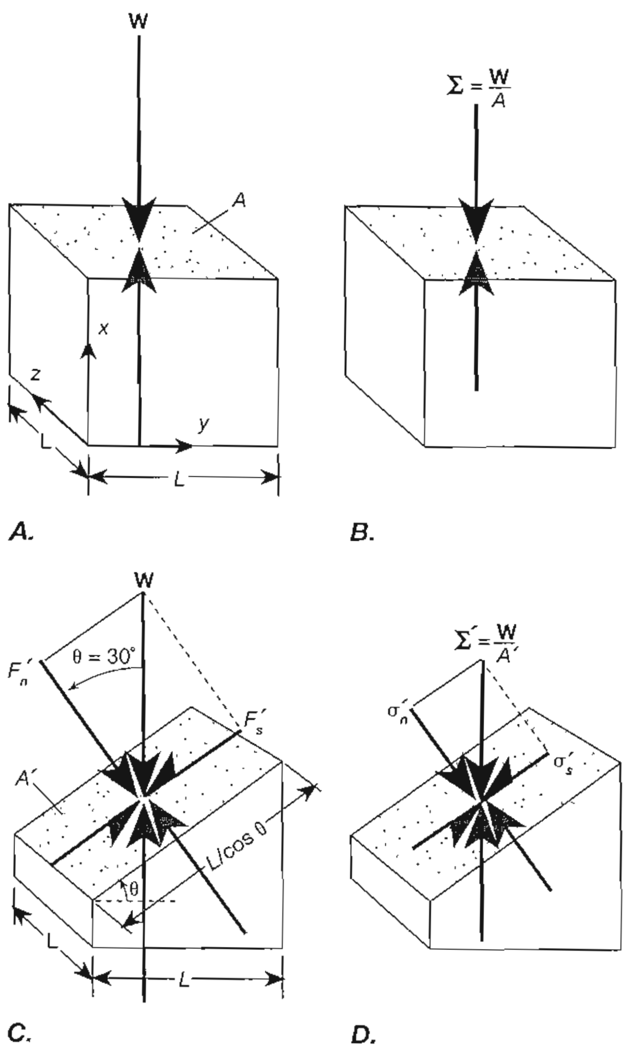


Figure 8.4 Determination of the stress components on surfaces of different orientation. *A*. In supporting the weight *W* of the tyranosaur in Figure 8.1A, the upper part of the pillar exerts a force W on the lower part across an arbitrary cross-section plane that has area *A* normal to the pillar axis. The lower part of the pillar exerts an equal and opposite force on the upper part. *B*. The magnitude of the surface stress Σ on the plane of area *A* is the force divided by the area. The force and the surface stress are normal to the surface, so there is no shear stress component. C. On a plane that is inclined at an angle $\theta = 30°$ through the pillar and has an area A', the same force W acts in the same direction. Components of the force normal and parallel to the plane are F_n and F_s. *D*. The magnitude of the surface stress Σ' is the same force W divided by the larger area A'. The magnitude of the normal stress and shear stress components σ'_n and σ'_s are the normal and parallel force components divided by A'.

the surface stress components according to the Mohr circle convention: compressive stress is positive.

$$\Sigma = \frac{W}{A} = \frac{80{,}000 \text{ N}}{2 \text{ m}^2} = 40{,}000 \text{ Pa} = 0.04 \text{ MPa} \qquad (8.5)$$

Here we are considering only the magnitudes of the vectors and surface stresses, so we do not use boldface type.

Because the force—and therefore the surface stress—acts exactly perpendicular to the area A, the normal stress component σ_n equals the magnitude of the surface stress Σ itself, and the shear stress component σ_s is zero. Thus,

$$\sigma_n = \frac{W}{A} = \Sigma = 40{,}000 \text{ Pa} \qquad \sigma_s = 0 \qquad (8.6)$$

Suppose, now, that we wanted to calculate the magnitude of the surface stress Σ' acting on a plane in this same column that is inclined at an angle $\theta = 30°$ to the left and has an area A' (Figure 8.4C). We have,

$$\Sigma' = \frac{W}{A'} = \frac{W}{A/\cos\theta} = \frac{80{,}000 \text{ N}}{2.309 \text{ m}^2} \approx 34{,}640 \text{ Pa}$$
$$= 0.03464 \text{ MPa} \qquad (8.7)$$

where the areas A and A' are related by

$$A = LL \qquad A' = L(L/\cos\theta) = A/\cos\theta \qquad (8.8)$$

Notice from Equation (8.7) that although the weight is the same, the magnitude of the surface stress on A' is smaller than that on A, because the area A' is larger than A (Figure 8.4D). The force components normal and parallel to the inclined plane, F'_n and F'_s, are

$$F'_n = W\cos\theta \qquad F'_s = W\sin\theta \qquad (8.9)$$

and the normal stress and shear stress components σ'_n and σ'_s are simply the corresponding force components divided by the area across which they act:

$$\sigma'_n = \frac{F'_n}{A'} = \frac{W\cos\theta}{A'} = \Sigma'\cos\theta$$
$$= 30{,}000 \text{ Pa} = 0.03 \text{ MPa} \qquad (8.10)$$

$$\sigma'_s = \frac{F'_s}{A'} = \frac{W\sin\theta}{A'} = \Sigma'\sin\theta$$
$$= 17{,}320 \text{ Pa} = 0.01732 \text{ MPa} \qquad (8.11)$$

We can relate the components σ'_n and σ'_s on the surface A' to the normal stress component σ_n on the surface A. In Equations (8.10) and (8.11), we write the area A' in terms of A using Equation (8.8) and then we use Equation (8.6) to obtain

$$\sigma'_n = \frac{W\cos\theta}{A'} = \frac{W\cos\theta}{A/\cos\theta} = \sigma_n\cos^2\theta \qquad (8.12)$$

$$\sigma'_s = \frac{W\sin\theta}{A'} = \frac{W\sin\theta}{A/\cos\theta} = \sigma_n\sin\theta\cos\theta \qquad (8.13)$$

This example shows that neither the traction nor the surface stress (a pair of equal and opposite tractions) is a vector quantity because they are both inseparable from the area, and thus the orientation, of the surface on which they act. The transformation equations relating the normal and shear components of the surface stress on two differently oriented planes (Equations 8.12 and 8.13) are very different from those relating the normal and tangential components of the force vector (Equations 8.9). Equations (8.12) and (8.13) include the transformation equations for the force vector (the numerators) as well as the equations accounting for the change in area with orientation (the denominators). These two effects result in products and squares of sine and cosine functions rather than just the first-order terms in these functions as in Equations (8.9).

Thus a traction has the characteristics of a vector only if we consider a surface of fixed orientation. The force vector, however, is independent of the orientation of the surface on which it acts. This difference is the most important distinction between traction, or stress, and force.

The Two-Dimensional Stress at a Point

We know the stress $\boldsymbol{\sigma}$ at a point in a body if we can determine the normal stress and shear stress components—written (σ_n, σ_s)—that act on a plane of *any* orientation passing through that point. There are, of course, an infinite number of such planes, so we need to know what minimum amount of information enables us to determine the stress components on any plane.

For the two-dimensional case, if the normal stress components on the planes are either all compressive or all tensile, the stress is particularly easy to visualize because it can be represented by an ellipse. If we plot all possible surface stresses as pairs of arrows from a common origin, the ends of the arrows fall on an ellipse called the **stress ellipse** (Figure 8.5A). States of stress are possible in which some normal stresses are compressive and some are tensile; in this situation, the stress ellipse is not defined. We concentrate on the intuitively simpler case in which the stress ellipse is a complete representation of the state of stress σ at a point in a body.

In general, the surface stresses are not perpendicular to the planes on which they act. Thus both the normal stress and the shear stress components (σ_n, σ_s) on an arbitrary surface are nonzero. The only exceptions are the surface stresses that are parallel to the major and minor axes of the ellipse (see the caption that accompanies Figure 8.6A). These two surface stresses are the principal stresses[5] $\hat{\sigma}_1$ and $\hat{\sigma}_3$ (Figure 8.5A). The planes on which the principal stresses act are the **principal planes**, and coordinate axes parallel to the prin-

[5] Here and throughout the book, we use "hats" (circumflexes) above symbols to indicate principal values or principal coordinates.

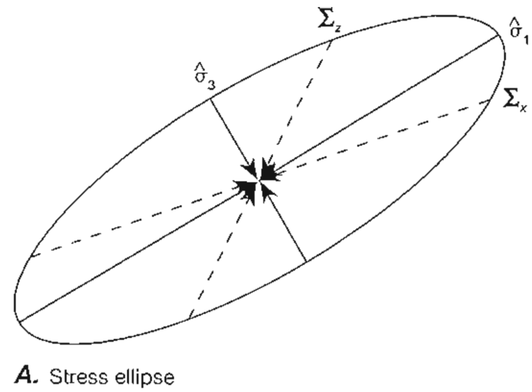
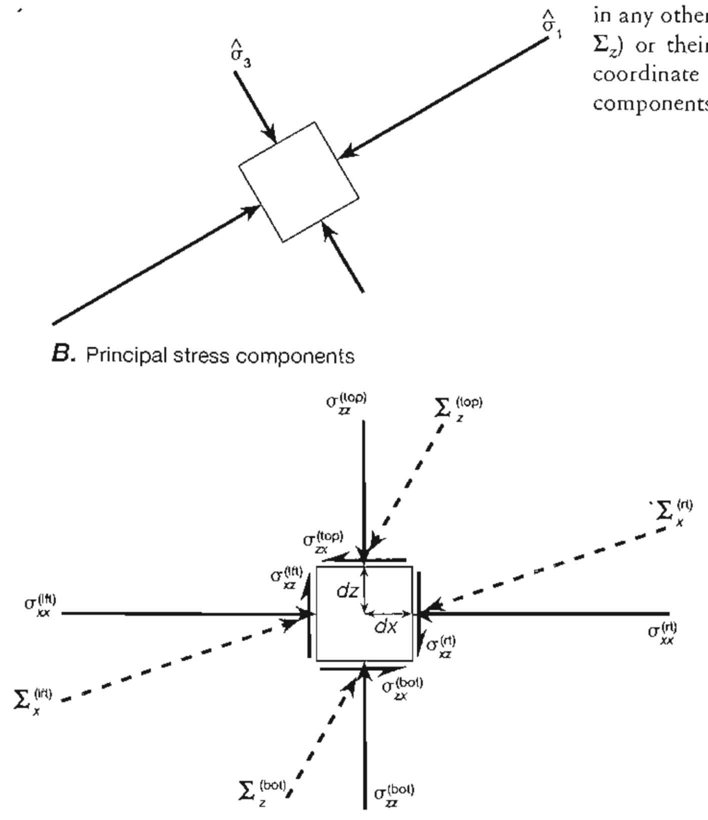
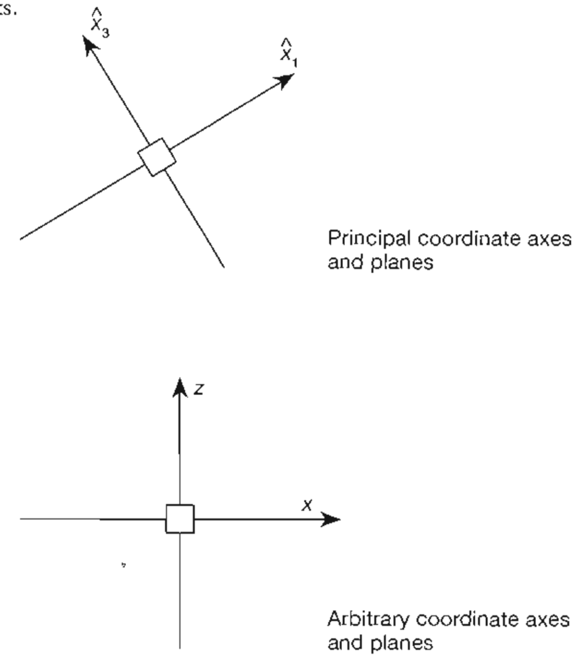


Figure 8.5 Representation of the state of two-dimensional stress at a point. *A*. The stress ellipse is the locus of all the surface stresses (plotted as pairs of arrows) that act on planes of all orientations through the point. In two dimensions, those planes are all normal to the plane of the diagram. All normal stress components must have the same sign in order to define a stress ellipse. The principal stresses are the major and minor axes of the ellipse and are the maximum and minimum of all normal stresses on the planes. The principal coordinate axes are parallel to the principal stresses, and the planes normal to the principal stresses are the principal planes. (See the legend accompanying Figure 8.6*A* for other details.) *B*. The stress at a point can be completely defined in the principal coordinate system ($\hat{x}_1$, $\hat{x}_3$) by the two principal stresses ($\hat{\sigma}_1$, $\hat{\sigma}_3$) that act on the two perpendicular principal coordinate planes. We represent the point by an infinitesimal coordinate square. *C*. The stress at a point can also be completely defined in any other coordinate system (x, z) by specifying the surface stresses (Σ_x, Σ_z) or their components (σ_{xx}, σ_{xz}), (σ_{zz}, σ_{zx}) on the two perpendicular coordinate planes. Superscripts identify specific tractions and traction components.





cipal stresses are the principal coordinates or principal axes[6] $\hat{x}_1$ and $\hat{x}_3$. The principal stresses are the maximum and minimum of all the surface stresses acting on planes of any orientation through the point, and by convention we label them such that

$$\hat{\sigma}_1 \geq \hat{\sigma}_3 \tag{8.14}$$

The magnitudes and orientations of the principal stresses $\hat{\sigma}_1$ and $\hat{\sigma}_3$ completely define the stress ellipse—and therefore the stress $\boldsymbol{\sigma}$ at a point.

The principal stresses are perpendicular to the principal planes on which they act, so the shear stress components on the principal planes are zero. Thus the magnitudes of the principal surface stresses are completely defined by their normal stress components $\hat{\sigma}_1$ and $\hat{\sigma}_3$. The converse is also true: Any plane on which the shear stress is zero (such as plane *A* in Figure 8.4*A*) must be a principal plane, and the normal stress on that plane must be a principal stress. Because the shear stresses are zero on the principal planes, using the principal stresses is a particularly simple way to define the stress at a point.

[6] The principal coordinates are labeled $\hat{x}_1$ and $\hat{x}_3$ instead of x and z so that they are directly associated with the principal stresses $\hat{\sigma}_1$ and $\hat{\sigma}_3$, respectively, to which they are parallel. The practice of distinguishing coordinate axes by different subscripts is a common one which we use specifically to label the components of the stress tensor. We describe the notation below and in more detail in Section 8.4.

We represent a point in a two-dimensional body as an infinitesimally small square of material. Opposite sides of the square represent the opposite sides of a plane through the point; and the perpendicular pairs of sides therefore represent two perpendicular planes through the point. Figure 8.5*B* shows the coordinate square in the principal coordinate system, with the principal stresses $\hat{\sigma}_1$ and $\hat{\sigma}_3$, the principal axes $\hat{x}_1$ and $\hat{x}_3$, and the principal planes, which are the sides of the square.

One surface stress does not define the complete state of stress at a point, as is evident from the fact that the length of one of the surface stresses in the stress ellipse does not define the complete shape of the ellipse. The two principal stresses completely define the shape of the stress ellipse. The surface stresses that act on *any* two perpendicular planes through the point, however, also completely define the shape of the stress ellipse.

We define general coordinates x and z perpendicular to the sides of a square of any specified orientation. We refer to planes *perpendicular* to x as x planes, and we refer to planes *perpendicular* to z as z planes. We can then label each stress component according to both the plane on which it acts and the coordinate to which it is parallel (Figure 8.5*C*). The components of the surface stress Σ_x acting on the x plane of the coordinate square are σ_{xx} and σ_{xz}. The first subscript x shows that both components act on the x plane; the second subscript shows that the components are parallel to the x and z coordinates, respectively. Thus σ_{xx} is the normal stress component, and σ_{xz} is the shear stress component (in Figure 8.5*C*, each of the tractions and traction components is labeled individually). Similarly, for the surface stress Σ_z acting on the z plane, σ_{zz} is the normal stress component, and σ_{zx} is the shear stress component.

Thus the stress $\boldsymbol{\sigma}$ at a point is completely defined either by the principal stresses ($\hat{\sigma}_1$, $\hat{\sigma}_3$) and their orientations, or, in the x–z coordinate system, by the surface stresses Σ_x and Σ_z or their components (σ_{xx}, σ_{xz}) and (σ_{zx}, σ_{zz}).

$$\boldsymbol{\sigma} = \begin{cases} \hat{\sigma}_1 \\ \hat{\sigma}_3 \end{cases} \quad \text{or} \quad \boldsymbol{\sigma} = \begin{cases} \Sigma_x: & (\sigma_{xx}, \sigma_{xz}) \\ \Sigma_z: & (\sigma_{zz}, \sigma_{zx}) \end{cases} \tag{8.15}$$

The only case for which one surface stress is sufficient to define the stress at a point is for a hydrostatic pressure, in which case the stress ellipse is a circle.

We require that the coordinate square be in mechanical equilibrium, which means that its acceleration parallel to each of the coordinate axes must be zero and that its angular acceleration must be zero. Thus both the forces and the moments of these forces acting on the square must sum to zero.

We know from Equation (8.4) that the normal traction and shear traction components on opposite sides of a plane must be equal and opposite (Figure 8.2*B*). Accordingly, from (Figure 8.5*C*),

$$\begin{aligned} \sigma_{xx}^{(\mathrm{rt})} &= -\sigma_{xx}^{(\mathrm{lft})} & \sigma_{zz}^{(\mathrm{top})} &= -\sigma_{zz}^{(\mathrm{bot})} \\ \sigma_{xz}^{(\mathrm{rt})} &= -\sigma_{xz}^{(\mathrm{lft})} & \sigma_{zx}^{(\mathrm{top})} &= -\sigma_{zx}^{(\mathrm{bot})} \end{aligned} \tag{8.16}$$

The product of a traction component and the area on which it acts defines a force component acting on the coordinate square. Using Figure 8.5*C*, we sum all the force components that are parallel to the x axis, and separately we sum all force components that are parallel to the z axis. Thus we require

$$\begin{aligned} \| x: &\quad \sigma_{xx}^{(\mathrm{rt})}A_x + \sigma_{xx}^{(\mathrm{lft})}A_x + \sigma_{zx}^{(\mathrm{top})}A_z + \sigma_{zx}^{(\mathrm{bot})}A_z = 0 \\ \| z: &\quad \sigma_{zz}^{(\mathrm{top})}A_z + \sigma_{zz}^{(\mathrm{bot})}A_z + \sigma_{xz}^{(\mathrm{rt})}A_x + \sigma_{xz}^{(\mathrm{lft})}A_x = 0 \end{aligned} \tag{8.17}$$

If we use Equations (8.16) to eliminate one of each of the traction component pairs from Equations (8.17), we obtain the identity $0 = 0$. Thus Equations (8.16) are the conditions that must be met if the forces are to sum to zero.

Taking moments of the forces about the origin—or, in essence, about the y axis—involves only the shear traction components, because the moment arms for the normal traction components are all zero. From Figure (8.5*C*), the infinitesimal dimensions of the square are $2dx$ and $2dz$, whereby taking all the moments and requiring their sum to be zero gives

$$\begin{aligned} \sigma_{xz}^{(\mathrm{rt})}A_x dx + \sigma_{xz}^{(\mathrm{lft})}A_x(-dx) + \sigma_{zx}^{(\mathrm{top})}A_z dz \\ + \sigma_{zx}^{(\mathrm{bot})}A_z(-dz) = 0 \end{aligned} \tag{8.18}$$

Because $A_x = A_z$ and $dx = dz$, we can eliminate these quantities from the equation by division and, using Equation (8.16), show that the shear tractions, and therefore the shear stresses, are related, respectively, by

$$\sigma_{xz}^{(\mathrm{lft})} = -\sigma_{zx}^{(\mathrm{bot})} \quad \text{and} \quad \sigma_{xz} = -\sigma_{zx} \tag{8.19}$$

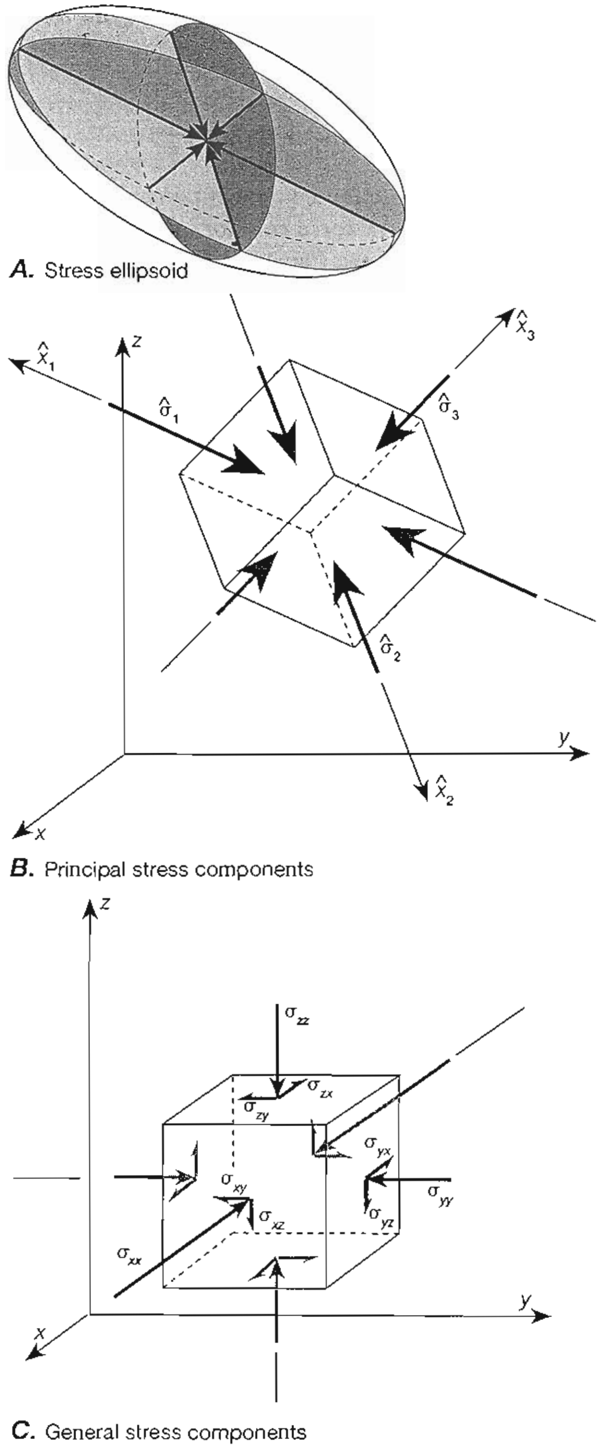
Thus of the four stress components in the x–z coordinate system (second equation 8.15), only three are independent: σ_{xx}, $\sigma_{xz} = -\sigma_{zx}$, and σ_{zz}.

The Three-Dimensional Stress at a Point

The description of the stress in three dimensions is a direct extrapolation of its description in two dimensions. In the simple case for which all normal stress components have the same sign, the stress at a point is represented by a stress ellipsoid (Figure 8.6*A*). The major, intermediate, and minor principal axes of the ellipsoid are parallel to the principal coordinate axes. They represent the maximum, intermediate, and minimum principal stresses, respectively, which we label in accordance with the convention

$$\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \hat{\sigma}_3 \tag{8.20}$$

The principal stresses are the surface stresses acting on



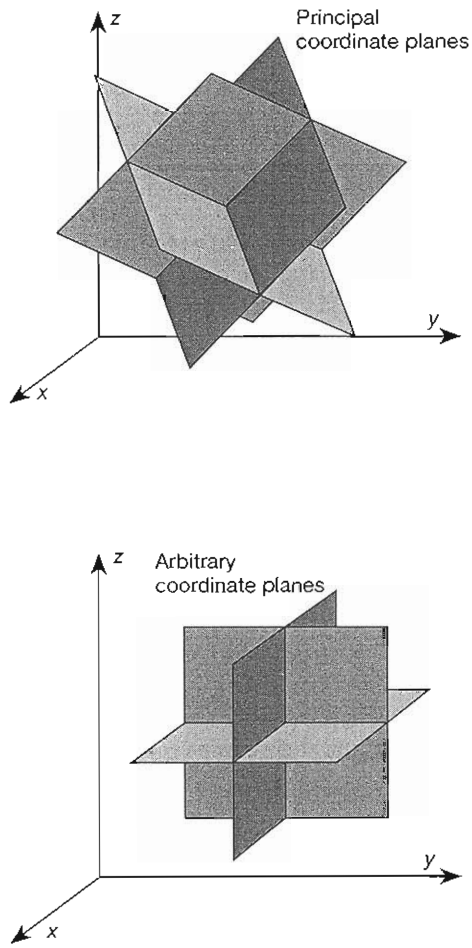


Figure 8.6 The state of three-dimensional stress at a point. *A*. The stress ellipsoid is defined by the surface stress(es) that act on planes of all possible orientations through a point. Shaded planes are the principal planes. Stress components are the principal stresses. For the representation of stress to be an ellipsoid, the normal components of the surface stress(es) must all be either compressive or tensile. The orientation of the plane on which any particular surface stress(es) acts is not immediately obvious from the stress ellipsoid. The components of the outward unit normal vector n ro the plane, however, are $(n_1, n_2, n_3) = (\hat{\Sigma}_1^{(n)}/\hat{\sigma}_1, \hat{\Sigma}_2^{(n)}/\hat{\sigma}_2, \hat{\Sigma}_3^{(n)}/\hat{\sigma}_3)$; where $(\hat{\Sigma}_1^{(n)}$, $\hat{\Sigma}_2^{(n)}$, and $\hat{\Sigma}_3^{(n)})$ are the components of the parricular surface stress parallel ro the rhree principal axes of the ellipsoid, and $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}_3$ are the principal stresses that parallel those three axes. *B*. Representation of the stress in principal coordinates. C. Representation of the stress in general coordinates.

the three mutually perpendicular principal planes through a point. On the principal planes, normal stresses have extreme values and shear stresses are zero. We represent the point as an infinitesimal cube whose faces are parallel to the principal planes and perpendicular to the principal axes $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ (Figure 8.6*B*).

The stress ellipsoid can also be defined by the surface stresses Σ_x, Σ_y, and Σ_z and their components that act on *any* three mutually perpendicular planes through a point (Figure 8.6C). The coordinate axes x, y, and z are parallel to the mutual intersections of those three planes, and the point is represented as an infinitesimal cube whose faces are parallel to the three planes. The surface stress acting on each face of the cube has three components, one parallel to each of the coordinate axes (compare Figure 8.1.3). One component is a normal stress; the other two are shear stresses. Each stress component is composed of a pair of equal and opposite traction components that act on opposite faces of the cube. We label the components of these three surface stresses by using the same convention we used for two-dimensional stress. Each component with two identical subscriprs is a normal stress; each component with rwo different subsctipts is a shear stress.

Thus the stress ellipsoid, which defines the stress at a point, is uniquely described by the three principal stresses $\hat{\sigma}_1$, $\hat{\sigma}_2$, and $\hat{\sigma}_3$ and their orientations (Figure 8.6B) or by three surface stresses Σ_x, Σ_y, and Σ_z or their components acting on three mutually perpendicular surfaces through the point (Figure 8.6C).

$$\sigma = \begin{cases} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{cases} \quad \text{or} \quad \sigma = \begin{cases} \Sigma_x: & (\sigma_{xx}, \sigma_{xy}, \sigma_{xz}) \\ \Sigma_y: & (\sigma_{yy}, \sigma_{yx}, \sigma_{yz}) \\ \Sigma_z: & (\sigma_{zz}, \sigma_{zx}, \sigma_{zy}) \end{cases} \qquad (8.21)$$

In general, nine components are required to define the three-dimensional stress at a point. Of these nine components, only six are independent however, because the moments of the forces acting on the cube taken about each coordinate axis must sum to zero, giving (compare Equations 8.18, and 8.19)

$$\sigma_{xy} = -\sigma_{yx} \qquad \sigma_{xz} = -\sigma_{zx} \qquad \sigma_{yz} = -\sigma_{zy} \qquad (8.22)$$

In order for us to analyze a problem as a two-dimensional case, the plane in which the problem is analyzed must be a principal plane containing two principal stresses—for example, $\hat{\sigma}_1$ and $\hat{\sigma}_3$—and it must be perpendicular to the third principal stress—for example, $\hat{\sigma}_2$.

Stress Tensor Notation

In continuum mechanics, the stress is defined by a mathematical quantity called the **stress tensor**, which we discuss in Section 8.4 (see Box 8.2). The srress tensor components have the same numerical values as the Mohr circle stress components, but the signs of the components are determined by a different convention, and in particular, the signs of the shear stress components may be different. To distinguish the components of the stress tensor from the Mohr circle stress components discussed above, we label the three orthogonal coordinate axes x_1, x_2, and x_3 instead of x, y, and z, and we label the components of the stress tensor with numerical subscripts. The first subscript is the number of the coordinate axis that is perpendicular to the plane on which the stress component acts, and the second subscript is the number of the coordinate axis that is parallel ro the stress component.[7] For example, the stress component σ_{13}, is a shear stress component that acts on the x_1 plane (first subscript) and is parallel to the x_3 axis (second subscript). The component σ_{11} is a normal stress component that acts on the x_1 plane (first subscript) and is parallel to the x_1 axis (second subscript). As before, the normal stress components have two identical subscrips, and the shear stress components have two different subscripts.

The components of the stress tensor are written in a specific order to form a **matrix**. The surface stresses that act on the three coordinate surfaces are written in a column, in ordet of increasing subscript from top to bottom. The components for each of those surface stresses are then written in a row, the rows being in the same order as the surface stresses. Thus the first subscript is the same in each row, and it increases in each column from top to bottom; the second subscript increases in each row from left to right. The components of the three-dimensional stress tensor are written in principal coordinates $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ or in general coordinates x_1, x_2, and x_3, respectively, as

$$\sigma = \begin{bmatrix} \hat{\Sigma}_1 \\ \hat{\Sigma}_2 \\ \hat{\Sigma}_3 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_1 & 0 & 0 \\ 0 & \hat{\sigma}_2 & 0 \\ 0 & 0 & \hat{\sigma}_3 \end{bmatrix} \quad \text{or}$$

PRINCIPAL DIAGONAL

$$\sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad (8.23)$$

PRINCIPAL DIAGONAL

The normal stress components appear along the principal diagonal of the matrix, and the shear stress components appear in the off-diagonal positions. With the tensor sign convention, Equation (8.22) becomes

$$\sigma_{12} = \sigma_{21} \qquad \sigma_{13} = \sigma_{31} \qquad \sigma_{23} = \sigma_{32} \qquad (8.24)$$

The three relationships in Equation (8.24) define the symmetry of the stress tensor, a term that refers to the equality of the shear stress components that occur, in the matrix, in symmetric positions relative to the principal diagonal, as in Equation (8.23). Note that with this sign convention, the symmetrically related shear stress components are *equal,* not opposite (compare Equation 8.22), even though one is a clockwise and the other a counterclockwise shear stress (Figure 8.6C). The notation for the principal stresses is unchanged, but the matrix shows explicitly that all the shear stress components associated with the principal srresses are zero.

In two dimensions, we have only two coordinate directions, which we generally take either to be $\hat{x}_1$ and $\hat{x}_3$ or to be x_1 and x_3, in which case the x_1–x_3 plane must be perpendicular to the intermediate principal axis $\hat{x}_2$. Thus the state of two-dimensional stress is specified only by the two surface stresses that act on the two

[7] Using two different notations to distinguish between the Mohr circle sign convention and the tensor sign convention for the stress components is a convenience we adopt for this book; it is not a distinction that is generally observed.

coordinate surfaces, and each surface stress has only two components. The matrix representing the two-dimensional stress at a point therefore has only four components.

$$\sigma = \begin{bmatrix} \hat{\Sigma}_1 \\ \hat{\Sigma}_3 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_1 & 0 \\ 0 & \hat{\sigma}_3 \end{bmatrix} \quad \text{or}$$

$$\sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix} \tag{8.25}$$

Here again the matrix is symmetric because $\sigma_{13} = \sigma_{31}$.

We use the stress tensor notation throughout this book in applications of stress to the study of deformation in the Earth. Table 8.2 presents a convenient reference to the notation we use for discussing stress.

8.3 The Mohr Diagram for Two-Dimensional Stress

For simplicity, we restrict our discussion in this section to two-dimensional stress. The extension to three-dimensional stress is discussed in Box 8.4.

The stress ellipse indicates that the normal stress and shear stress components on a plane must change progressively with the orientation of the plane. The relationship between the orientation of the plane and the values of normal stress and shear stress on the plane is difficult to extract from the stress ellipse (see the legend that accompanies Figure 8.6*A*). That relationship is remarkably simple, however, when the stress is plotted

Table 8.2 Notation for Stress[a]

Σ	Traction or surface stress acting on a planar surface of specified orientation.
σ_n, σ_s	Normal and shear components respectively for both the surface stress and the traction.
Σ_x, Σ_y, Σ_z Σ_k	Surface stresses or tractions acting on the coordinate surfaces that are normal, respectively, to the coordinate axes *x, y, z* (or to axes x_k, where *k* takes the values 1, 2, and 3).
σ	The stress at a point: a second-rank symmetric tensor quantity.
$\hat{\sigma}_k$	Principal stresses (maximum, intermediate, and minimum for $k = 1$, 2, and 3, respectively), which are normal stress components acting on coordinate planes in the principal coordinate system $\hat{x}_k$. The shear stress components on these planes are zero. Because these values are the lengths of the principal axes of the stress ellipsoid, they define the stress σ at a point.
σ_{xx}, σ_{xy}, σ_{xz} σ_{yx}, σ_{yy}, σ_{yz} σ_{zx}, σ_{zy}, σ_{zz}	Mohr circle stress components defining the stress $\boldsymbol{\sigma}$ at a point in the (*x, y, z*) coordinate system. Each row contains the components, one of the surface stresses, or tractions, $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$, $\boldsymbol{\Sigma}_z$, respectively. The first subscript is the axis normal to the coordinate plane on which the component acts; the second subscript is the axis parallel to the stress component. In defining $\boldsymbol{\sigma}$, compressive normal stress and counterclockwise shear stress components are positive, tensile normal stress and clockwise shear stress components are negative.
$\sigma_{k\ell}$	Components the stress $\boldsymbol{\sigma}$ at a point. These components are the same as the components for the three surface stresses or tractions $\boldsymbol{\Sigma}_k$ that act on the three coordinate surfaces, referred to the $\boldsymbol{x}_k$ coordinate system. For each value of subscript $k = 1, 2$, and 3, subscript ℓ takes on the values 1, 2, 3, which indicate the three components of each surface stress; x_k is normal to the coordinate surface on which a component acts, and x_ℓ is parallel to the direction of the component. Normal components have $k = \ell$; shear components have $k \neq \ell$. These components differ from the Mohr circle stress components only in the sign convention. For the geologic tensor sign convention, tensor components have the same sign as the traction components that act on the *negative* side of the coordinate surface.
${}_D\sigma$	The differential stress. A positive scalar quantity equal to the difference between the maximum and minimum principal stresses.
$\bar{\sigma}_n$	The mean normal stress: the average of the normal stress components of the stress tensor in any coordinate system. It is a scalar invariant of the stress tensor.
${}_\Delta\sigma_{k\ell}$	The deviatoric stress components, equal to the stress tensor components with the mean stress subtracted from each of the normal stress components.
${}_E\sigma_{k\ell}$	The effective stress components, equal to the components of the stress tensor with the pore fluid pressure subtracted from each of the normal stress components.

[a] Boldface type, either with or without subscripts, indicates vectors and tensors; normal type with subscripts indicates scalar components of vectors and tensors. We use the same notation for the traction and its components as for the stress at a point and its components, even though the stress is actually defined by the pair of equal and opposite tractions acting on opposite sides of the surface. The sign of a component may be different, depending on whether it is a traction component or a component of the stress, although its absolute value is the same. Where the distinction is important, the context makes the intent clear. In some cases, however, we specify a particular traction by using additional superscripts whose meaning is self-evident.

on a Mohr diagram[8] for which the horizontal axis is the value of the normal stress σ_n, and the vertical axis is the value of the shear stress σ_s.

For a given stress, we can show (see Section 8.5) that on the Mohr diagram, the normal stress and shear stress components on planes of all possible orientations through a point plot on a circle called the Mohr circle. The center of the circle lies on the normal stress axis. As before, compressive normal stresses and counterclockwise shear couples are considered positive. Characteristics of the Mohr circle show clearly how the stress at a point is related to the surface stresses on planes through the point. We number these characteristics to provide a convenient means of referencing them in subsequent sections.

1) *The Mohr Diagram*

(i) The diagram has axes that are values of stress. It is therefore very important to distinguish the Mohr diagram from a diagram of physical space, whose axes are spatial coordinates. It is always necessary to draw a separate diagram of physical space, along with the Mohr diagram, and to transfer data carefully from one diagram to the other (Figure 8.7).

[8] Named after Christian Otto Mohr (1835–1918), a German professor of mechanics and civil engineering.

(ii) The Mohr circle is a complete representation of the stress at a point, because the normal stress and shear stress components of the surface stress on planes of all possible orientations through the point are included on the circle. Each point on the circle represents the surface stress on a different plane.

2) *Principal Stresses*

(i) The maximum and minimum normal stresses have values defined by the intersection of the Mohr circle with the σ_n axis (Figure 8.7*B*). Note that these two points are the only surface stresses on the Mohr circle for which the shear stress is zero.

3) *Surface Stress and the Orientation of Planes*

(i) The orientation of a plane in physical space is defined relative to known coordinate axes by the orientation of its *normal* n, not the orientation of the plane itself (Figure 8.7*A*). For example, the angle θ in physical space (Figure 8.7*A*) is measured between $\hat{x}_1$ and the normal n to a plane *P*. θ is also the angle between the normal stress components on the $\hat{x}_1$ coordinate plane (σ_1) and on the plane *P* ($\sigma_n^{(P)}$) because $\hat{\sigma}_1$ is parallel to x_1 and $\sigma_n^{(P)}$ is parallel to *n*.

(ii) Angles measured in physical space are doubled when plotted on the Mohr diagram. Angles are measured in the same sense on the Mohr diagram as in

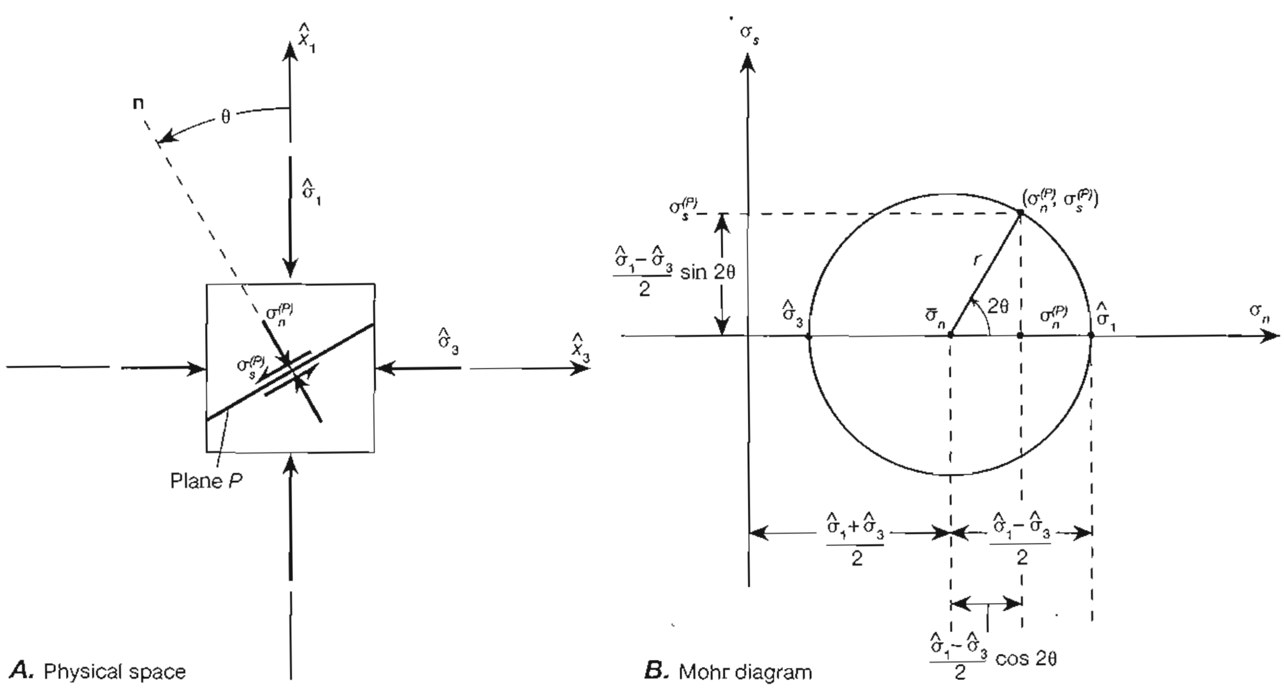


Figure 8.7 Plotting on a Mohr diagram the stress at a point. *A*. Relationships in physical space among the stress components, the principal coordinate axes, and the plane *P* with its normal n. *B*. Stress at a point represented on a Mohr diagram by the Mohr circle. Superscripts (*P*) identify stress components acting on a plane *P*.

physical space. As θ takes on values from 0° to 180°, the angle plotted on the Mohr diagram 2θ takes on values between 0° and 360°, and the entire Mohr circle is swept out (Figure 8.7*B*). All planes have two normals which are 180° apart. Therefore in physieal spaee, the angles $180° \leq \theta < 360°$ are redundant because they merely duplicate the orientations of the plane defined by the angles $0° \leq \theta < 180°$.

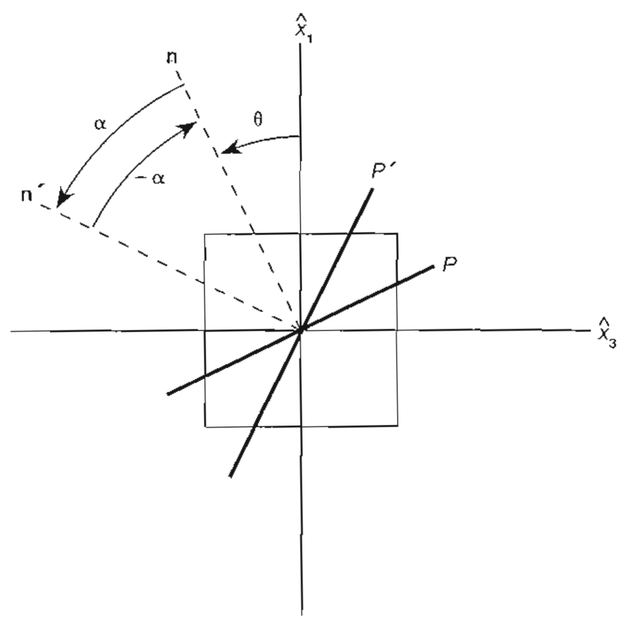
(iii) The normal stress and shear stress components $(\sigma_n^{(P)}, \sigma_s^{(P)})$ acting on a plane *P* have a simple relationship on the Moht diagram to the orientation in physical space of the normal n to the plane. In physical space, suppose that n (or $\sigma_n^{(P)}$) makes an angle θ from the axis of maximum principal stress $\hat{x}_1$ (or $\hat{\sigma}_1$) (Figure 8.7*A*). On the Mohr circle, the surface stress components on plane *P*, $(\sigma_n^{(P)}, \sigma_s^{(P)})$, plot at the end of the radius that lies at the angle 2θ from the radius to the maximum principal stress $(\hat{\sigma}_1, 0)$ (Figure 8.7*B*).

(iv) If there are two arbitrary planes in physical space *P* and *P'* whose normals are n and n' (Figure 8.8*A*), and if the angle from $\hat{x}_1$ to n is a counterclockwise angle θ and the angle from n to n' is a counterclockwise angle α, then on the Mohr diagram there are two points on the Mohr circle, $(\sigma_n^{(P)}, \sigma_s^{(P)})$ and $(\sigma_n^{(P')}, \sigma_s^{(P')})$, that define the normal stress and shear stress components on *P* and *P'*, respectively (Figure 8.8*B*). The angle between radii to those points is 2α, measured countetclockwise from $(\sigma_n^{(P)}, \sigma_s^{(P)})$ to $(\sigma_n^{(P')}, \sigma_s^{(P')})$. If the angle in physical space is measured from n' to n, it is clockwise and therefore a negative angle $-\alpha$, in which case a clockwise (negative) angle -2α is plotted on the Mohr circle from the radius at $(\sigma_n^{(P')}, \sigma_s^{(P')})$ to the radius at $(\sigma_n^{(P)}, \sigma_s^{(P)})$.

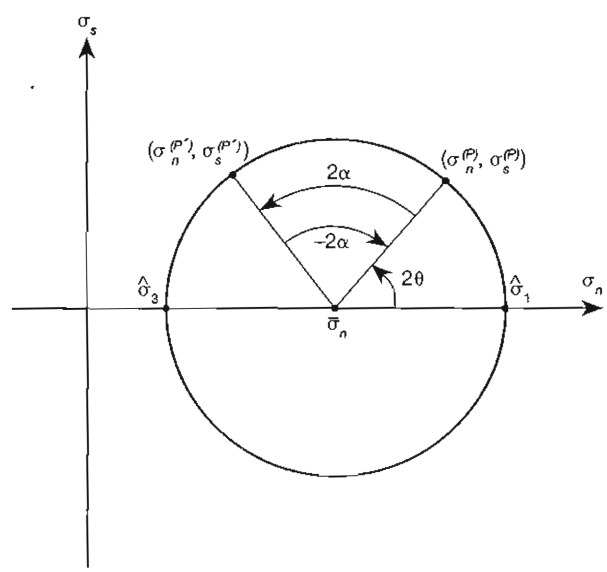
(v) The surface stress components that lie at opposite ends of any diameter of the Mohr circle ($2\alpha = 180°$) are the components acting on perpendicular planes in physical space ($\alpha = 90°$) (Figure 8.9*A*, *B*). Thus the principal stresses $\hat{\sigma}_1$ and $\hat{\sigma}_3$, which act on perpendicular planes, plot at opposite ends of a diameter of the circle, as do the two pairs of components $(\sigma_{xx}, \sigma_{xz})$ and $(\sigma_{zz}, \sigma_{zx})$ that specify the surface stresses acting on the perpendicular coordinate planes of an arbitrary coordinate system. Fundamentally, this statement is a corollary to the fact that angles measured in physical space are doubled when plotted on the Mohr diagram (item (ii) above).

4) *Conjugate Planes of Maximum Shear Stress*

(i) The stresses on the planes whose normals lie at $\theta = \pm 45°$ to the maximum principal stress $\hat{\sigma}_1$ in physical space (Figute 8.10*A*) occur on the Mohr circle

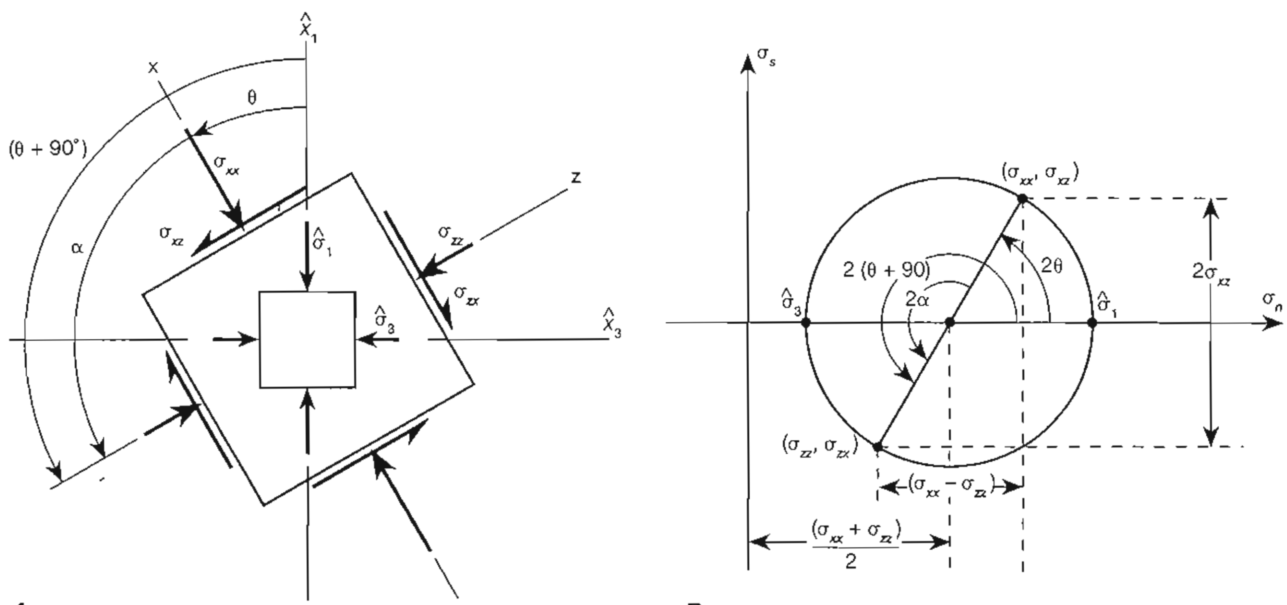


A. Physical space



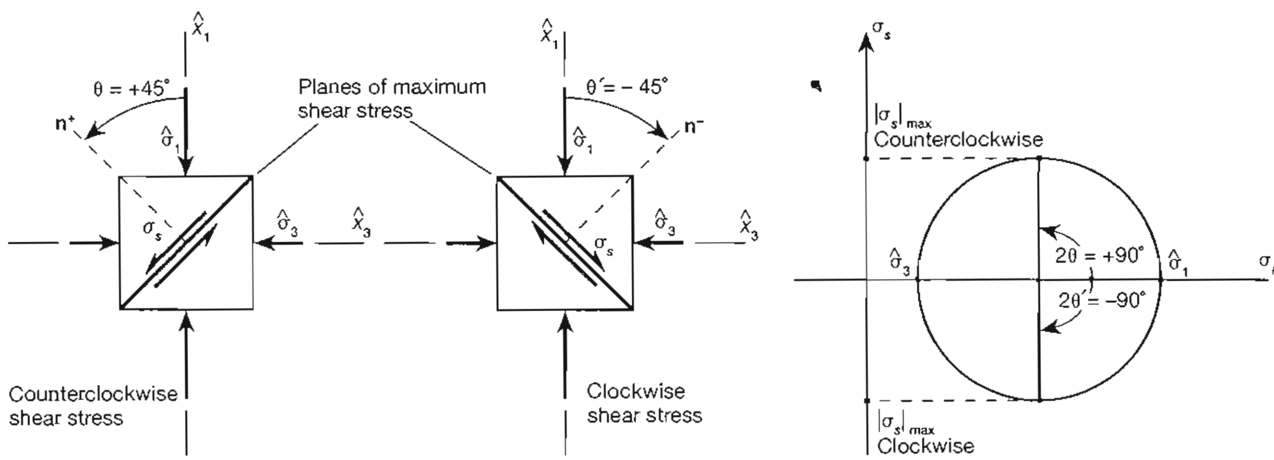
B. Mohr diagram

Figure 8.8 The geometric relationship berween planes in physical space and the stress components on those planes. *A.* The orienrations of the planes *P* and *P'* in physical space are determined by the orientarions of their normals n and n', respectively. *B.* The geomerry on rhe Mohr diagram of rhe srress componenrs acting on the planes shown in part *A.* Note rhat the angles plorred are double rhe angles measured berween normals to the planes in physical space bur rhar the sense of rotation in measuring rhe angles is rhe same.



A. Physical space **B.** Mohr diagram

Figure 8.9 Transferring stress components from a diagram of physical space to a Mohr diagram. A. Diagram of physical space showing srress components in two coordinate systems ($\hat{x}_1$, $\hat{x}_3$) and (x, z). The different sets of stress components represent the same stress and are shown on different sized coordinate squares for convenience. B. Representation on a Mohr diagram of the principal stresses as well as the stress components in the general coordinate system shown in part A. Components of surface stress acting on two planes that are perpendicular to each other in physical space plot on the Mohr circle at opposite ends of a diameter. The two scalar invariants of the stress are the center of the Mohr circle, defined by the mean of the normal stresses at opposite ends of a diameter, and by the length of the diameter, which is related by the Pythagorean theorem to the sum of the squares of $2\sigma_{xz}$ and $(\sigma_{xx} - \sigma_{zz})$.



A. Physical space **B.** Mohr diagram

Figure 8.10 Planes of maximum shear stress. A. Relationships in physical space between the planes of maximum absolute shear stress aud the principal stresses. The two planes are said to be coujugate shear planes. B. Mohr diagram showing the plot of the maximum absolute shear stresses and their relatiouships to the principal stresses.

at $2\theta = \pm 90°$, measured from $(\hat{\sigma}_1, 0)$ (Figure 8.10*B*). On these planes, the absolute value of the shear stress $|\sigma_s|$ is a maximum. These planes are the conjugate planes of maximum shear stress, and in physical space, the planes themselves lie at $\pm 45°$ to the maximum compressive stress $\hat{\sigma}_1$. The stresses on these planes plot at opposite ends of a diameter of the Mohr circle, and therefore in physical space the normals to the planes are perpendicular, as are the planes themselves.

5) *Scalar Invariants of the Stress*

(i) The magnitude of the stress at a point is nniquely-characterized by two scalar invariants of the stress, which are defined by the location of the center of the circle $\overline{\sigma}_n$, called the mean normal stress, and by the radius of the circle r, which is also the maximum possible absolute value of the shear stress, $|\sigma_s|_{(\max)}$ (Figure 8.7*B*). These two qnantities are, respectively, half the sum and half the difference of the principal stresses.

$$\overline{\sigma}_n = \frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2} \qquad r = |\sigma_s|_{(\max)} = \frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2} \qquad (8.26)$$

$\overline{\sigma}_n$ and r are called scalar invariants because they are scalars whose values are the same for any set of components $(\sigma_{xx}, \sigma_{xz})$, $(\sigma_{zz}, \sigma_{zx})$ that define the same stress. In other words, if we know the end points of any diameter of the Mohr circle, we can construct the whole circle, because $\overline{\sigma}_n$ and r can always be determined. For the end points of an arbitrary diameter $(\sigma_{xx}, \sigma_{xz})$, $(\sigma_{zz}, \sigma_{zx})$ (Figure 8.9*B*),

$$\begin{aligned} \overline{\sigma}_n &= \frac{\sigma_{xx} + \sigma_{zz}}{2} \\ r &= 0.5[(\sigma_{xx} - \sigma_{zz})^2 + (2\sigma_{xz})^2]^{0.5} \end{aligned} \qquad (8.27)$$

The second Equation (8.27) results from setting up a right triangle in Figure 8.9*B* with sides parallel to the axes and the diameter of the Mohr circle as the hypotenuse and then applying the Pythagorean theorem to calculate the diameter, which is twice the radius. In principal coordinates, the normal stresses become the principal stresses and the shear stresses are zero, so Equations (8.27) reduce to Equations (8.26). Our ability to construct the entire Mohr circle knowing only the surface stress components at the two end points of one diameter shows that the stress is completely defined by the surface stress components on any two perpendicular planes.

The scalar invariants of the stress describe fundamental geometric characteristics of the stress ellipse (Figure 8.5*A*). The mean normal stress $\overline{\sigma}_n$ is proportional to the mean radius of the ellipse, and the square of the radius of the Mohr circle, r^2, is proportional to the area of the ellipse.

6) *Equations of the Mohr Circle*

(i) The formulas for calculating the normal stress and shear stress components ou any plane in physical space whose normal **n** is at an angle θ from the maximum principal stress $\hat{\sigma}_1$ are easily determined from the geometry of the Mohr circle (Figure 8.7*B*).

$$\begin{aligned} \sigma_n &= \overline{\sigma}_n + r \cos 2\theta \\ &= \left[\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2}\right] + \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right] \cos 2\theta \\ \sigma_s &= r \sin 2\theta = \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right] \sin 2\theta \end{aligned} \qquad (8.28)$$

Note that Equations (8.28) are written in terms of the scalar invariants of the stress.

The Mohr circle provides a very quick and convenient method for obtaining solutions to stress problems, and in Appendix 8.1, we give examples of some of the problems that can be solved by this means. Because it provides a simple way to visualize the stress at a point, we will use the Mohr circle repeatedly in our applications of stress to understanding brittle deformation in rocks.

8.4 The Stress Tensor

The stress at a point $\boldsymbol{\sigma}$ belongs to a group of mathematical quantities called second rank tensors, and it is therefore called the stress **tensor** (see Box 8.2). A second rank tensor can always be represented by a matrix of nine numbers in three dimensions, such as in Equation (8.23), or by a matrix of four nnmbers in two dimensions, such as in Equation (8.25).

Although the stress components $(\sigma_{xx}, \sigma_{xz})$, $(\sigma_{zz}, \sigma_{zx})$ that we defined in Section 8.2 are required for Mohr circle problems, the sign of the shear components is in fact not precisely defined. For example, a shear stress that is counterclockwise when viewed from one direction is clockwise when viewed from the opposite direction. Thus its sign in the Mohr circle convention depends on the direction from which it is viewed. This ambiguity does not cause us difficulty in solving two-dimensional Mohr circle problems, as long as we interpret the solution to a stress problem by using the same diagram of physical space in which the problem was defined. The ambiguity is intolerable, however, for more complex problems and for mathematical computation. In order to have an unambiguous definition of sign, we must use the set of stress components that represent the stress tensor (see Box 8.2).

To distinguish the components of the stress tensor from the Mohr circle stress components, we use a

Box 8.2 What Is a Tensor?

A tensor is a mathematical quantity that can be used to describe the state or the physical properties of a material. We represent a tensor by a set of scalar components referred to a particular coordinate system. Tensor components must change in a prescribed way if the coordinate axes are rotated (see Box 8.1 for this effect in vectors).

The **rank** of a tensor indicates how many scalar components are required to describe it completely. The number of components c equals the dimension d of the physical space raised to the power given by the rank r.

$$c = d^r$$

In three-dimensional space ($d = 3$), for example, a **scalar** is a tensor of zero rank ($r = 0$) and so has $3^0 = 1$ component. Common examples include temperature, mass, and volume. Scalars are defined simply by their magnitude and are invariant under a change of coordinates. We represent them mathematically by a single symbol, such as T for temperature and m for mass.

A **vector** is a first-rank tensor ($r = 1$) with $3^1 = 3$ components in three-dimensional space ($d = 3$). Force, velocity, and acceleration are all vector quantities. Vectors describe physical quantities that are characterized by magnitude and a single direction. The values of the vector components change under a rotation of coordinates as prescribed by Equations 8.1.6. The components are represented mathematically by a symbol with a single subscript, such as F_k. The subscript k is understood to take on the values 1, 2, and 3 in three-dimensional space and the subscripted symbol represents the three components (F_1, F_2, F_3), each of which is parallel to one of the coordinate axes. In two-dimensional space, k takes on just two values, such as 1 and 2.

A second-rank tensor ($r = 2$) in three-dimensional space ($d = 3$) has $3^2 = 9$ components; the most important examples of these in structural geology are stress, introduced in this chapter, and strain, introduced in Chapter 15. Second-rank tensors are used to describe physical quantities that have magnitudes and are associated with two directions. For the stress tensor, for example, the two directions associated with each component are the orientation of the normal to the plane on which the stress component acts and the orientation of the stress component acting on that plane. The transformation equations for two of the components of the second-rank stress tensor are given in essence by Equations (8.36) for transformation from principal coordinates and by Equations (8.3.3) and (8.3.4) for transformation from general coordinates. Second-rank tensors, such as the stress $\sigma_{k\ell}$, are represented by a symbol with two subscripts. For three-dimensional space, both k and ℓ independently take on the values 1, 2, and 3. Thus for each value that k can have, ℓ can take on any of its values, thereby providing distinct symbols for each of the nine components. In two dimensions, k and ℓ take on only two values each, such as 1 and 2.

Note that for vectors, the terms in the transformation equations (Equations 8.1.6) involve the first power of the sine and cosine functions, whereas for second-rank tensors, the terms in the transformation equations (Equations 8.36, 8.3.3, and 8.3.4) involve the products of sine and cosine functions. The difference is due to the fact that transformation of vector components involves transformation of a single direction, whereas transformation of second-rank tensor components involves transformation of two directions. Thus tensors of different rank are characterized by different types of transformation equations for their components.

The magnitude of any physical quantity described by a tensor must be independent of the coordinate system in which we choose to describe it. Thus for each rank of tensor, there are a certain number of **scalar invariants** that define the magnitude of the quantity. For scalars, this fact is self-evident; the scalar is itself a magnitude and is invariant for any change of coordinate systems. Vectors have one scalar invariant, the magnitude, which we represent by the length of an arrow. This length is independent of the coordinate system in which we describe the vector. For second-rank tensors in three dimensions, three independent scalar invariants are needed to define the magnitude of the physical quantity; in two dimensions, a second-rank tensor has two scalar invariants. We discuss these invariants in Sections 8.3 and Box 8.4 as properties 5.i and 5.ii respectively.

Physical quantities that are described by tensors of higher rank also exist. For example, the piezoelectric material constants are represented by a third-rank tensor whose components can be symbolized by A_{ijk}, and the elastic constants of a material are defined, in general form, by a fourth-rank tensor symbolized by $A_{ijk\ell}$. These tensors are associated with three and four directions, respectively. In particular, the piezoelectric material constants describe the relationship between the stress on a material (two directions) and the associated electric field (one direction). The elastic constants describe the relationship between the stress on a material (two directions) and the associated strain (two directions). These higher-rank tensors, do not concern us in this book.

slightly different notation, which is explained at the end of Section 8.2. As a short hand notation for the stress tensor components, we refer to the three coordinate axes collectively as x_k, where the subscript k can take on the value 1, 2, or 3. The nine stress components are written collectively as $\sigma_{k\ell}$, where for each value of $k = 1, 2,$ or 3, ℓ can take on the value 1, 2, or 3 (compare Equation 8.23).

The surface stress on any of the coordinate surfaces consists of a pair of equal and opposite tractions or, equivalently, sets of equal and opposite traction components, which we represent as acting on opposite faces of an infinitesimal cube. We define the positive sides of the cube to be the ones facing in a positive coordinate direction, and we define the negative sides to be the ones facing in a negative coordinate direction (Figure 8.11*A*). The values of the stress tensor components are then defined to be equal either to the traction components acting on the negative cube faces, which gives the geologic sign convention, or to the traction components acting on the positive cube faces, which gives the engineering sign convention.[9] We use the same symbol for both the traction components and the stress tensor components, becanse they have the same absolute value and differ only in sign convention. As noted before, we will distinguish the traction components, where it is necessary, by using appropriate superscripts.

Using the geologic tensor sign convention, we see that the stress tensor component σ_{22} in Figure 8.11*B* is positive because the normal traction component acting on the negative side of the cube, $\sigma_{2\bar{2}}$, points in a positive coordinate direction. By the same token, σ_{22} in Figure 8.11C is negative because $\sigma_{2\bar{2}}$ points in a negative coordinate direction. Thus compressive states of stress are positive, and tensile states of stress are negative. This convention gives the same sign as the Mohr diagram sign convention that we defined for σ_{xx}, σ_{yy}, and σ_{zz} (Section 8.2), so there is no ambiguity for the normal stress components. If the coordinate axes x, y, and z are parallel to x_1, x_2, and x_3, respectively, then $\sigma_{xx} = \sigma_{11}$, $\sigma_{yy} = \sigma_{22}$, and $\sigma_{zz} = \sigma_{33}$.

The same argument yields the signs for the shear stress components. Figure 8.11*D* shows that σ_{23} and σ_{32} are *both positive* because the traction components acting on the negative side of the coordinate planes—$\sigma_{2\bar{3}}$ and $\sigma_{3\bar{2}}$, respectively—both point in positive coordinate directions. By the same token, Figure 8.11*E* shows that σ_{23} and σ_{32} are *both negative* because $\sigma_{2\bar{3}}$ and $\sigma_{3\bar{2}}$ both point in negative coordinate directions. Here we see the difference from the Mohr circle sign convention (Section 8.2). In Figure 8.11*D*, for example, even though both shear couples are positive in the tensor sign convention, σ_{23} is clockwise and σ_{32} is counterclockwise, which in the Mohr convention means negarive and positive signs respectively. In Figure 8.11*E* also, both shear couples are negative in the tensor sign convention, even though they are of opposite shear sense.

The tensor sign convention does not depend on the direction from which the diagram is viewed, and thus it is unambiguous.[10] In order to plot tensor components on a Mohr diagram, however, we must adopt special conventions to circumvent the ambiguity of sign for the shear stress components as defined for the Mohr diagram. We review these conventions in the next section.

Equations (8.24) show that the shear stress components in the stress tensor are not all indepeudent. These equations are equivalent to Equations (8.22), as we can show if we equate the axes x, y, and z with x_1, x_2, and x_3, and then when we change to the Mohr circle notation and sign convention. Because of Equations (8.24), the stress tensor $\boldsymbol{\sigma}$ is called a **symmetric** tensor of second rank (compare Box 8.2), and it is necessary to specify only six of the nine numbers in the matrix in order to define completely the stress at a point.

We can gain some appreciation for the significance of a second rank tensor by comparing it with a vector. A force is a vector quantity that has a directional quality and is represented by a row array of three scalars. A stress is a second rank tensor quantity that has a bidirectional quality and is represented by a column array of three surface stresses, each of which is in turn represented by a row array of three scalars. The three

[9] The engineering sign convention gives tensor stress components with the opposite sign from those given by the geologic sign convention. This convention is used in engineering and physics and for most analytic applications of continuum mechanics, so it is also common in the geologic literature. Thus for the stress components, we have four different sign conventions: the geologie and the engineering conventions for the Mohr diagram, and the geologic and the engineering conventions for the stress tensor. Unfortunately, all are found in the geologic literatnre, and to avoid confusion, one must always be careful to determine which convention is employed. Often the convention is not stated explicitly.

[10] A mathematically more precise method of defining the geologic tensor sign convention is with reference to the inward unit normal vectors, which are vectors of unit length that are normal to the cube faces and point inward toward the center of the cube. If on any particular face, the traction component aud the inward unit normal both point in positive coordinate directions or both point in negative coordinate directions, the stress tensor component is positive. If the component and the inward unit normal point in opposite coordinate senses, that is, one positive and the other negative, the stress tensor component is negative. This definition gives consistent results for any of the traction components on any face of the coordinate cube, so the sign need not be defined just in terms of the traction components on one side of the cube. For the engineering sign convention, the outward unit normal—that is, the unit normal veetor to the cube faces that points away from the center of the cube—is used as a reference.

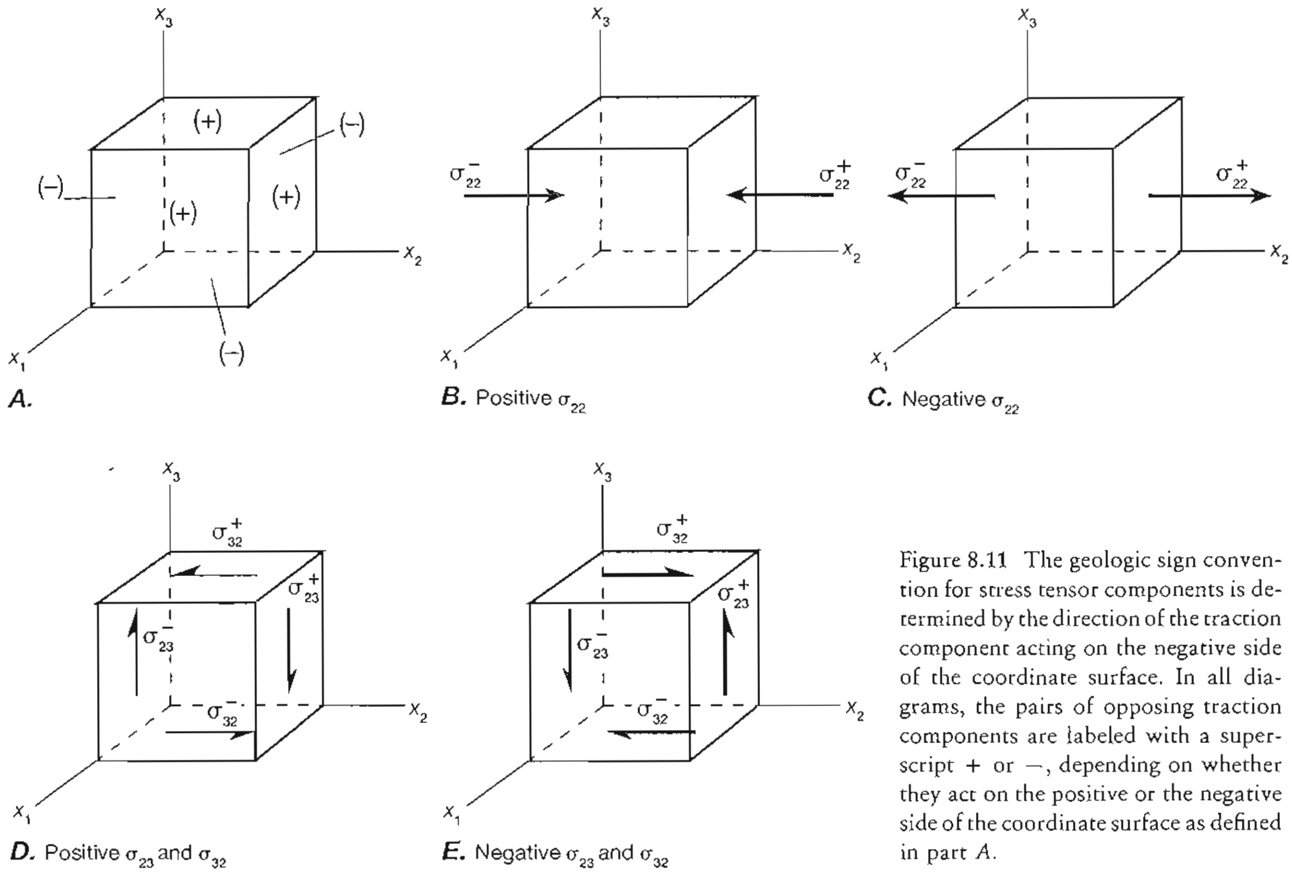


Figure 8.11 The geologic sign convention for stress tensor components is determined by the direction of the traction component acting on the negative side of the coordinate surface. In all diagrams, the pairs of opposing traction components are labeled with a superscript + or −, depending on whether they act on the positive or the negative side of the coordinate surface as defined in part *A*.

surface stresses are those that act on the three coordinate surfaces, and the two directions involved in each stress component are the orientation of the surface on which each stress component acts and the orientation of the stress component itself (see Box 8.2).

For a given stress at a point, the stress tensor $\boldsymbol{\sigma}$ is a fixed physical quantity described, for example, by the stress ellipsoid (Figure 8.6*A*). The *representation* of that tensor in components such as in the matrix in Equation (8.24), however, requires the specification of a particular coordinate system. In two differently oriented coordinate systems, the three surface stresses on the three coordinate planes are in general different, as can be seen from the stress ellipsoid. Thus the stress components (Equation 8.24) also generally have different values. Because the stress ellipsoid is the same, however, the stress at the point is the same, and it is always possible to calculate one set of components from another, given the angular relationships between the coordinate frames (see Section 8.5 and Box 8.3). The situation is analogous to that for the components of a vector as described in Box 8.1, but the equations we use to calculate one set of components from another for vectors are different from those we use for second-rank tensors (compare Equations 8.1.6 with 8.3.3 and 8.3.4).

For any vector, it is always possible to define a coordinate system in which all components of the vector are zero except one. This is the case when one coordinate axis is parallel to the vector. The analogous situation for the stress tensor, as for any symmetric second rank tensor, is that there always exists a coordinate system of a particular orientation for which all the shear stresses on all three coordinate surfaces are simultaneously zero, and the normal stresses on these coordinate surfaces are extrema—that is, maximum, minimum, or minimax[11] (Box 8.3). This special coordinate system is the principal coordinate system, and in these coordinates, the stress tensor is completely represented by the three normal stress components that are the principal stresses. The axes of the principal coordinate system are parallel to the principal axes of the stress ellipsoid (Figure 8.6*A*), and the principal stresses are the surface stresses parallel to those axes.[12]

[11] A minimax is a quantity that is a minimum in the plane that contains the maximum and the minimax and is a maximum in the perpendicular plane that contains the minimum and the minimax.

[12] For those familiar with linear algebra, the principal stresses and principal directions are the eigenvalues and eigenvectors, respectively, for the matrix of stress components.

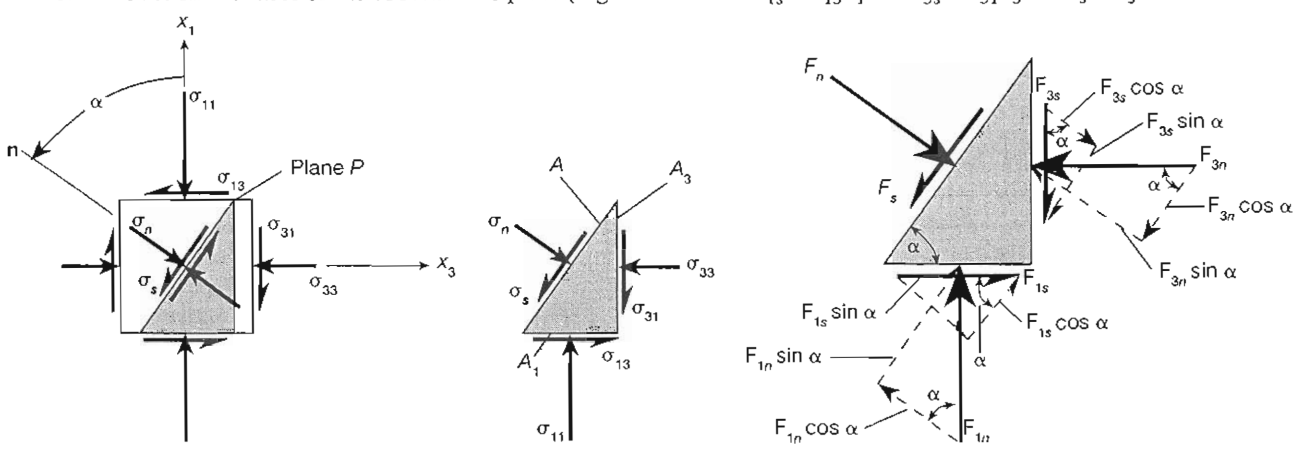
Box 8.3 Derivation of Principal Stresses in Two Dimensions

In order to show that principal stresses must exist for any stress tensor, we need to derive equations analogous to Equations (8.36) in terms of the stress components in a general coordinate system. The principles of the derivation are the same as those used to obtain Equations (8.36), and we merely outline the procedure here.

We limit ourselves to considering planes parallel to $\hat{x}_2$ so that our diagrams of physical space are only in the $\hat{x}_1$–$\hat{x}_3$ plane. We assume we know the orientation of the reference coordinate axes x_1 and x_3, both of which are normal to $\hat{x}_2$. Both normal and shear stresses act on the faces of the coordinate square (Figure 8.3.1*A*). The normal **n** to the plane *P* on which we determine the normal stress and shear stress components (σ_n, σ_s) makes an angle α with x_1. All stress components are drawn as positive components; α is drawn as a positive angle.

We isolate the shaded triangular element in Figure 8.3.1*B* and then construct a diagram of the forces acting on the triangular element (Figure 8.3.1*C*), where

$$\begin{array}{lll} F_{1n} = \sigma_{11}A_1 & F_{3n} = \sigma_{33}A_3 & F_n = \sigma_n A \\ F_{1s} = \sigma_{13}A_1 & F_{3s} = \sigma_{31}A_3 & F_s = \sigma_s A \end{array} \qquad (8.3.1)$$



A. Surface stress components ***B.*** Traction components ***C.*** Force components

Figure 8.3.1 Geometric relationships used to deduce the transformation equations for components of two-dimensional stress. *A*. An infinitesimal coordinate square in an arbitrary coordinate system, showing the components of stress on the coordinate surfaces and on an arbitrary plane *P*. All quantities are shown as positive quantities. *B*. The traction components acting on the exterior surfaces of the shaded triangle in part *A*. Tractions are labeled with the associated stress components, because we want the transformation equations in terms of stress. Sign differences are accounted for in formulating the equations. Areas *A*, A_1, and A_3 can be thought of as the areas of the sides of a triangular prism of unit dimension normal to the diagram. C. Forces acting on the isolated triangular element, showing their components parallel and perpendicular to plane *P*.

8.5 A Closer Look at the Mohr Circle for Two-Dimensional Stress

In this section, we derive the equations for the Mohr circle. From this derivation, the relationship between the stress tensor components and the Mohr circle becomes evident. We restrict the following discussion to two-dimensional problems. Manipulation of the equation for two dimensions is significantly less complex than for three dimensions, and retaining the third dimension adds little to intuitive understanding. The analysis in three dimensions proceeds along similar lines, as summarized in Box 8.4 (readers should finish this section before reading the box).

In order for us to analyze a stress problem in two dimensions, the third dimension must be parallel to one of the principal stresses. Because the principal stresses are mutually perpendicular, two of the principal stresses must then lie in the plane of the analysis. In terms of the stress ellipsoid (Figure 8.6*A*), the surface stresses in a two-dimensional stress analysis must all lie in one of the principal planes, and the planes on which they act are all parallel to the third principal stress.

Figure 8.12*A* shows the most common geometry for a two-dimensional analysis of stress. If $x_2 = \hat{x}_2$ (the intermediate principal stress axis), then $\sigma_{22} = \hat{\sigma}_2$, and the x_2 plane must be a principal plane. Thus the shear and where,

$$A_1 = A\cos\alpha \qquad A_3 = A\sin\alpha \tag{8.3.2}$$

We resolve each force vector into two components parallel to F_n and F_s, respectively, and then require equilibrium by setting the sum of the forces in each of these two directions equal to zero. Then, expressing forces in terms of stresses with Equations (8.3.1), rearranging the equations to isolate σ_n and σ_s on the left, substituting Equations (8.3.2) for A_1 and A_3, dividing through by A to eliminate it from the equations, and using the symmetry condition of the stress tensor Equations (8.24), we find that

$$\sigma_n = \sigma_{11}\cos^2\alpha - 2\sigma_{13}\sin\alpha\cos\alpha + \sigma_{33}\sin^2\alpha \tag{8.3.3}$$

$$\sigma_s = (\sigma_{11} - \sigma_{33})\sin\alpha\cos\alpha + \sigma_{13}(\cos^2\alpha - \sin^2\alpha) \tag{8.3.4}$$

We now wish to determine the orientation α_0 of planes on which σ_n is a maximum or a minimum. To this end, we differentiate Equation (8.3.3) with respect to α and set the result equal to zero.

$$\frac{d\sigma_n}{d\alpha} = 0 = (\sigma_{11} - \sigma_{33})\sin\alpha_0\cos\alpha_0 + \sigma_{13}(\cos^2\alpha_0 - \sin^2\alpha_0) \tag{8.3.5}$$

where we have used α_0 instead of α to indicate that the angle is no longer arbitrary. The right sides of Equations (8.3.5) and (8.3.4) are identical. This means that the condition for σ_n to be extreme is also the condition for σ_s to be zero.

We solve Equation (8.3.5) for α_0 by using the trigonometric identities:

$$\begin{aligned}\cos 2\alpha_0 &= \cos^2\alpha_0 - \sin^2\alpha_0\\ \sin 2\alpha_0 &= 2\sin\alpha_0\cos\alpha_0\\ \tan 2\alpha_0 &= \sin 2\alpha_0/\cos 2\alpha_0\\ \tan 2\alpha_0 &= \tan 2(\alpha_0 + 90^\circ)\end{aligned} \tag{8.3.6}$$

The result is

$$\tan 2\alpha_0 = \tan 2(\alpha_0 + 90^\circ) = \frac{-2\sigma_{13}}{\sigma_{11} - \sigma_{33}} \tag{8.3.7}$$

Thus Equations (8.3.5) shows that for planes on which σ_n is a maximum or a minimum, σ_s is zero. Equation (8.3.7) shows that there are two such planes. The normals to these planes make angles of α_0 and $(\alpha_0 + 90^\circ)$ with x_1. The planes are therefore perpendicular to each other (Figure 8.3.2), and their normals are the orientations of the principal axes of stress.

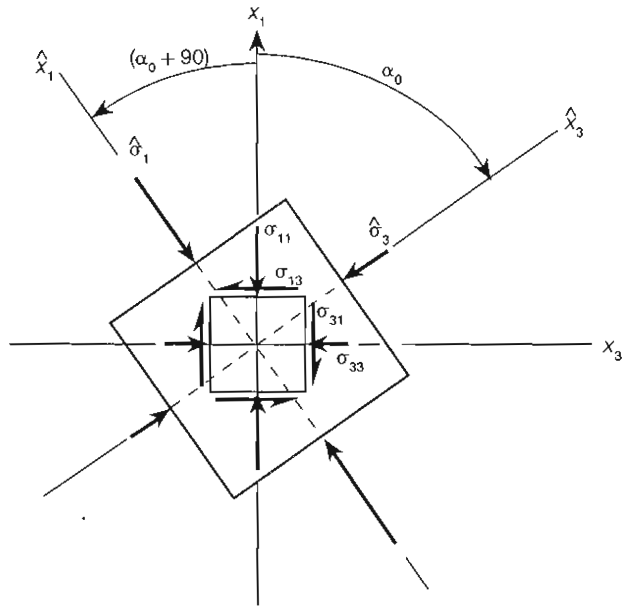


Figure 8.3.2 Stress components for a single stress shown on infinitesimal coordinate squares for a general coordinate system x_1–x_3 and for the principal coordinate system $\hat{x}_1$–$\hat{x}_3$. Both squares are infinitesimal, but they are drawn in different sizes for clarity. The angle α_0 is obtained from Equation (8.3.7).

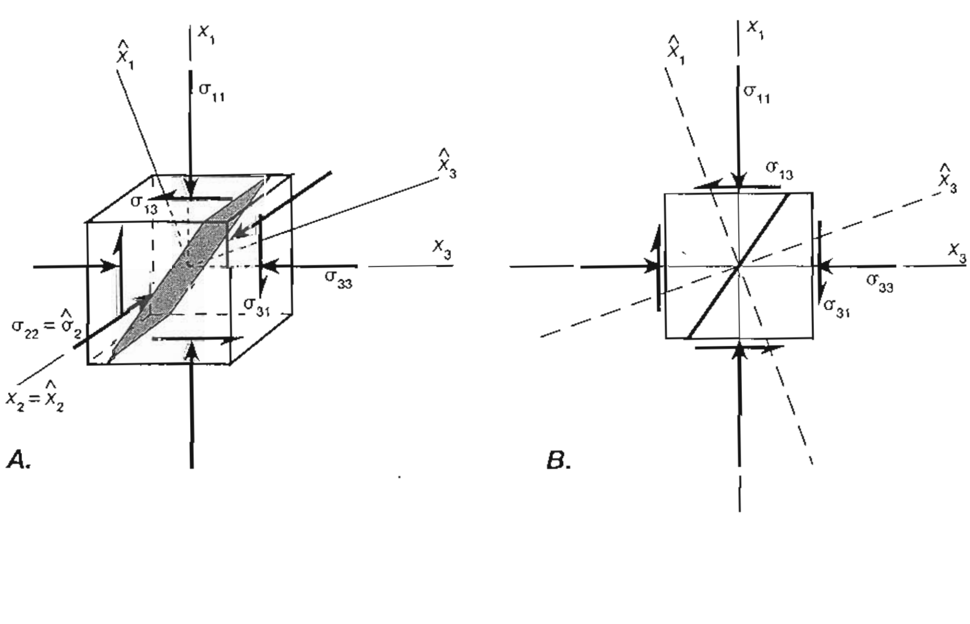


Figure 8.12 Geometry that permits a two-dimensional analysis of stress. *A.* If one coordinate axis, for example, x_2, is parallel to one of the principal coordinates, for example, $\hat{x}_2$, then the x_1–x_3 coordinate plane contains the principal axes $\hat{x}_1$ and $\hat{x}_3$, and the stresses can be analyzed in two dimensions in the x_1–x_3 plane. *B.* Appropriate two-dimensional diagram for analyzing the two-dimensional stress for the geometry shown in part *A.*

Box 8.4 The Mohr Diagram for Three-Dimensional Stress

In Section 8.5 we discuss determination of the surface stress acting on planes that are parallel to $\hat{x}_2$. The two-dimensional stress components are parallel to the $\hat{x}_1$–$\hat{x}_3$ plane. For the stresses in the other coordinate planes, exactly the same properties of the Mohr circle that are discussed in Section 8.5 apply. For planes parallel to any of the principal axes $\hat{x}_k$, a two-dimensional diagram of the $\hat{x}_i$–$\hat{x}_j$ plane is used, where $k \neq i < j \neq k$. Thus (i, j, k) can take on the values (1, 3, 2) (Figure 8.4.1*A*), (1, 2, 3) (Figure 8.4.1*B*), or (2, 3, 1) (Figure 8.4.1*C*). The general forms of the equations analogous to Equations (8.38) and (8.41) are

$$\text{for} \quad (i, j, k) = (1, 3, 2), (1, 2, 3), \text{ or } (2, 3, 1) \tag{8.4.1}$$

$$\sigma_n = \frac{\hat{\sigma}_i + \hat{\sigma}_j}{2} + \frac{\hat{\sigma}_i - \hat{\sigma}_j}{2} \cos 2\theta_k$$
$$\sigma_s = \frac{\hat{\sigma}_i - \hat{\sigma}_j}{2} \sin 2\theta_k \tag{8.4.2}$$

$$\left[\sigma_n - \left(\frac{\hat{\sigma}_i + \hat{\sigma}_j}{2}\right)\right]^2 + \sigma_s^2 = \left[\frac{\hat{\sigma}_i - \hat{\sigma}_j}{2}\right]^2 \tag{8.4.3}$$

where here θ_k is positive, measured counterclockwise about the $\hat{x}_k$ axis from $\hat{x}_i$ in the $\hat{x}_i$–$\hat{x}_j$ plane (Figure 8.4.2A). When $(i, j, k) = (1, 3, 2)$, we recover Equations (8.38) and (8.41).

To the main properties of a single Mohr circle discussed in Section 8.3, we append the following properties that apply to a Mohr diagram of three-dimensional stress.

1) THE MOHR DIAGRAM

(iii) The three-dimensional stress plots on a Mohr diagram as a set of three Mohr circles each of which is a graph of the surface stress components on sets of planes parallel to one of the principal axes (Figure 8.4.2). The three circles are defined by Equations (8.4.2), with Equation (8.4.1), and each involves one pair of the principal stresses. All the properties 1–6 discussed in Section 8.3 apply to each of these circles.

2) PRINCIPAL STRESSES

(iii) All three principal stresses plot on the σ_n axis. Each principal stress plots at a point that is common to two of the Mohr circles. If all the principal stresses are unequal, there are no other common points among the circles. Each of the principal stresses is at the opposite end of a Mohr circle diameter from each of the other two principal stresses, which is consistent with the fact that the three principal stresses in physical space act on three mutually perpendicular surfaces (cf. property 3v in Section 8.3).

3) SURFACE STRESS AND THE ORIENTATION OF PLANES

(vi) Planes that are not parallel to one of the principal axes have normals that do not lie in any of the principal coordinate planes (Figure 8.4.3*A*). The components of the surface stress on all such planes must plot on the Mohr diagram within the largest Mohr circle and outside the two smaller circles in the area shaded in Figure 8.4.2*B*. The construction on the Mohr diagram for determining the stress components on such a plane is indicated in Figure 8.4.3*B*, for which the geometry in physical space is shown in Figure 8.4.3*A*. The complexity of such three-dimensional problems is beyond the scope of this book, and our interest is confined to problems involving planes that parallel one of the principal axes.

4) CONJUGATE PLANES OF MAXIMUM SHEAR STRESS

(ii) The maximum absolute values of the shear stress on any plane in three-dimensional space plot on the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle at $2\theta_2 = \pm 90°$ (Figure 8.4.4*A*). These stresses occur on a conjugate set of planes in physical space that are parallel to $\hat{x}_2$ and that have normals lying in the $\hat{x}_1$–$\hat{x}_3$ plane at $\theta_2 = \pm 45°$ from $\hat{x}_1$ (Figure 8.4.4*B*). Thus although each Mohr circle individually has maximum absolute values of the shear stress (Figure 8.4.2*B*), the maxima for the $\hat{\sigma}_1$–$\hat{\sigma}_2$ and the $\hat{\sigma}_2$–$\hat{\sigma}_3$ Mohr circles are maxima only for the particular set of planes that are parallel to $\hat{x}_3$ and $\hat{x}_1$, respectively. The true maxima for planes of all possible orientations occur only at the maxima for the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle (Figure 8.4.4*A*).

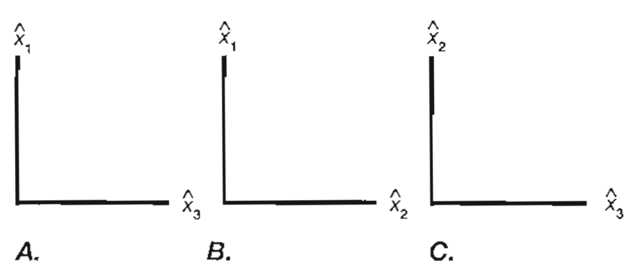


Figure 8.4.1 The pairs of principal coordinate axes for the three Mohr circles in three-dimensional stress. The axes must be oriented such that there is a clockwise rotation from the positive axis parallel to the larger normal stress toward the positive axis parallel to the smaller normal stress. This convention standardizes the change between Mohr circle convention and tensor sign convention for the shear stress components.

Box 8.4 *(continued)*

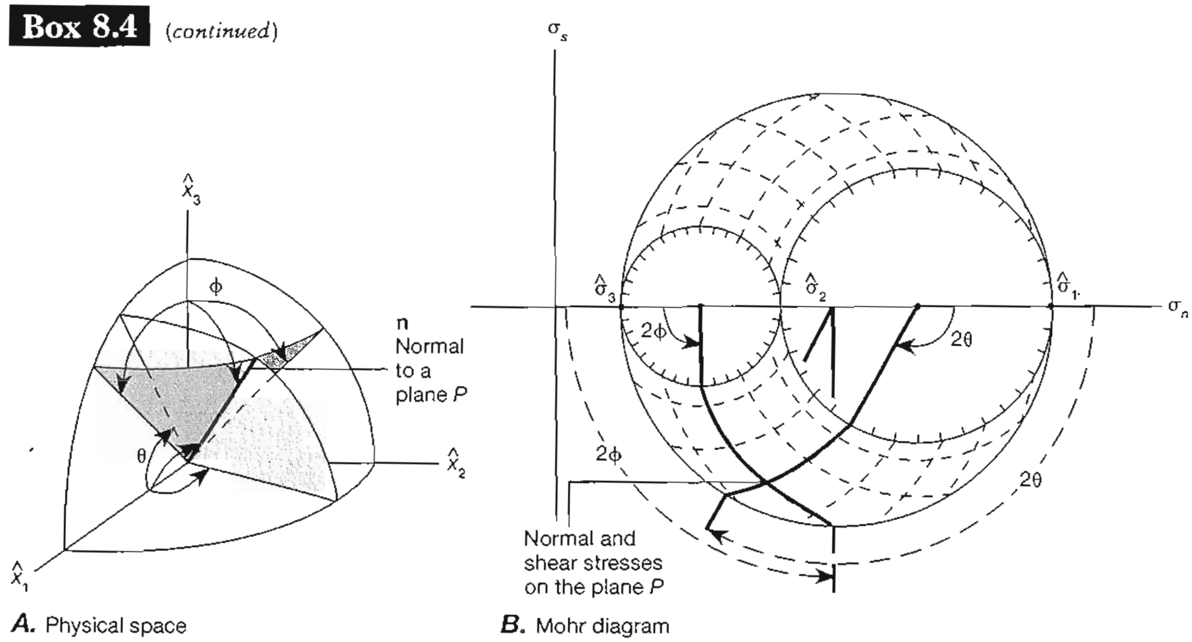


Figure 8.4.3 Mohr diagram for stress components on a plane of arbitrary orientation in three dimensions. *A*. Physical space: The normal n to the plane *P* (not shown) is defined by the angles θ from $\hat{x}_1$ and ϕ from $\hat{x}_3$. Counterclockwise angles measured in the principal coordinate planes are positive when the coordinate axes are viewed according to convention 1 in Section 8.5 (Figure 8.4.1). The θ is negative (clockwise) in both the $\hat{x}_1$–$\hat{x}_3$ plane and the $\hat{x}_1$–$\hat{x}_2$ plane; ϕ is positive (counterclockwise) in both the $\hat{x}_1$–$\hat{x}_3$ plane and the $\hat{x}_2$–$\hat{x}_3$ plane. *B*. Mohr diagram: The angles in part *A* are transferred to the Mohr diagram to determine the normal stress and shear stress acting on the plane *P*. Families of dashed curves are arcs concentric with the two smaller Mohr circles.

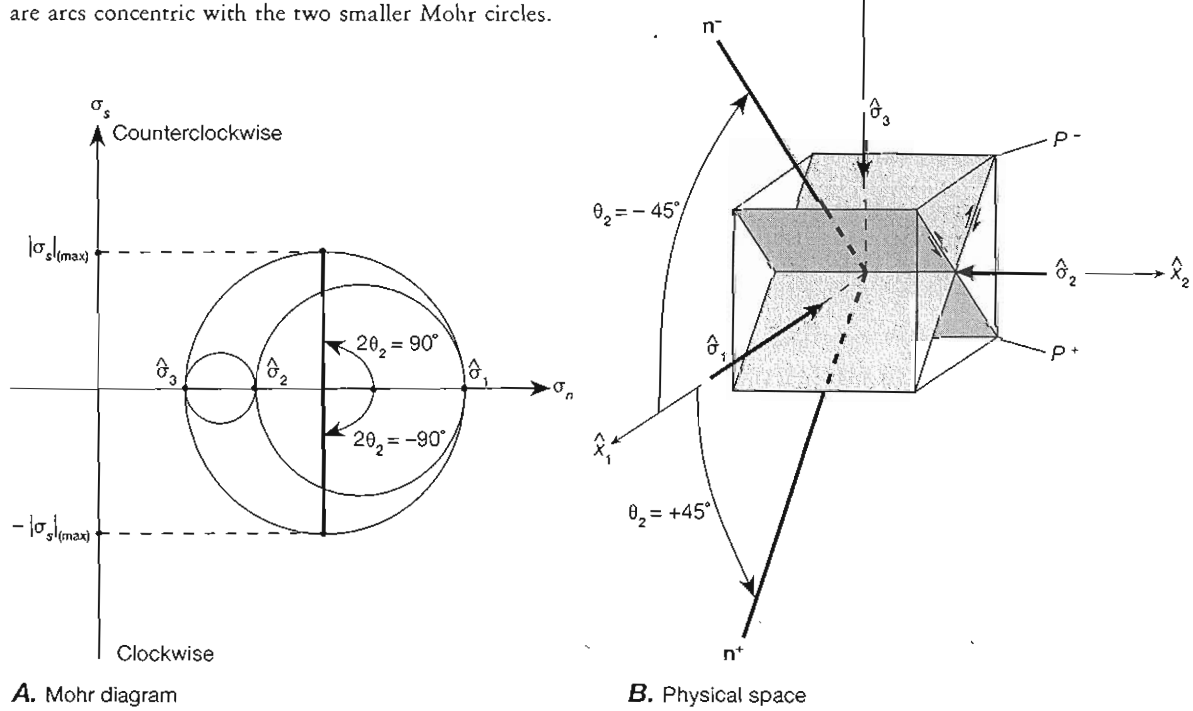


Figure 8.4.4 Planes of maximum shear stress in three dimensions. *A*. Mohr diagram showing maximum absolute values of the shear stress. *B*. Diagram of physical space showing the conjugate planes of maximum shear stress and their relationship to the principal stresses.

stress components on that plane must be zero ($\sigma_{21} = \sigma_{23} = 0$). The matrix of stress components must therefore include at least one row and, by symmetry of the stress tensor (Equations 8.24), one column in which the shear-stress components are zero.

$$\boldsymbol{\sigma} = \sigma_{k\ell} = \begin{bmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \hat{\sigma}_2 & 0 \\ \sigma_{31} & 0 & \sigma_{33} \end{bmatrix} \tag{8.29}$$

All the nonzero stress components are shown in Figure 8.12*A*, and all except $\sigma_{22} = \hat{\sigma}_2$ lie in the x_1–x_3 plane. Under these conditions, the surface stress on any plane parallel to $x_2 = \hat{x}_2$ is completely determined by the components of the stress tensor that lie in the x_1–x_3 plane (Figure 8.12*B*). Thus none of the stress components with a 2 in the subscript affects the stress on the planes parallel to $\hat{x}_2$. This fact justifies our use of a two-dimensional analysis.

The matrix of components for the most common two-dimensional stress tensor is obtained simply by eliminating from the matrix in Equation (8.29) all components having a 2 as one of the subscripts, leaving

$$\boldsymbol{\sigma} = \sigma_{k\ell} = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix} \tag{8.30}$$

In order to derive the relationship among the normal stress and shear stress components and the orientation of the plane on which they act, we pose the following question: If we know the orientation of the principal axes, and we know the values of the principal stresses at a point, how can we determine the components of the surface stress that act on a plane of arbitrary orientation through that point? Consider the infinitesimal cube (Figure 8.13*A*) centered on the origin of the principal axes with faces parallel to the principal planes. The plane *P* is parallel to $\hat{x}_2$ but is otherwise of arbitrary orientation. With this geometry we can use a two dimensional analysis to determine the surface stress on *P*. The two dimensional stress tensor expressed in principal coordinates (the first Equation 8.25) is obtained from the first Equation (8.23) by deleting all stress components that have a 2 as a subscript.

$$\boldsymbol{\sigma} = \hat{\sigma}_{k\ell} = \begin{bmatrix} \hat{\sigma}_1 & 0 \\ 0 & \hat{\sigma}_3 \end{bmatrix} \tag{8.31}$$

Figure 8.13*B* is the diagram of the stress components.

The following conventions, which we used in constructing Figure 8.13*B*, are crucial for establishing a consistent relationship between the stress tensor components and the values plotted on a Mohr diagram.

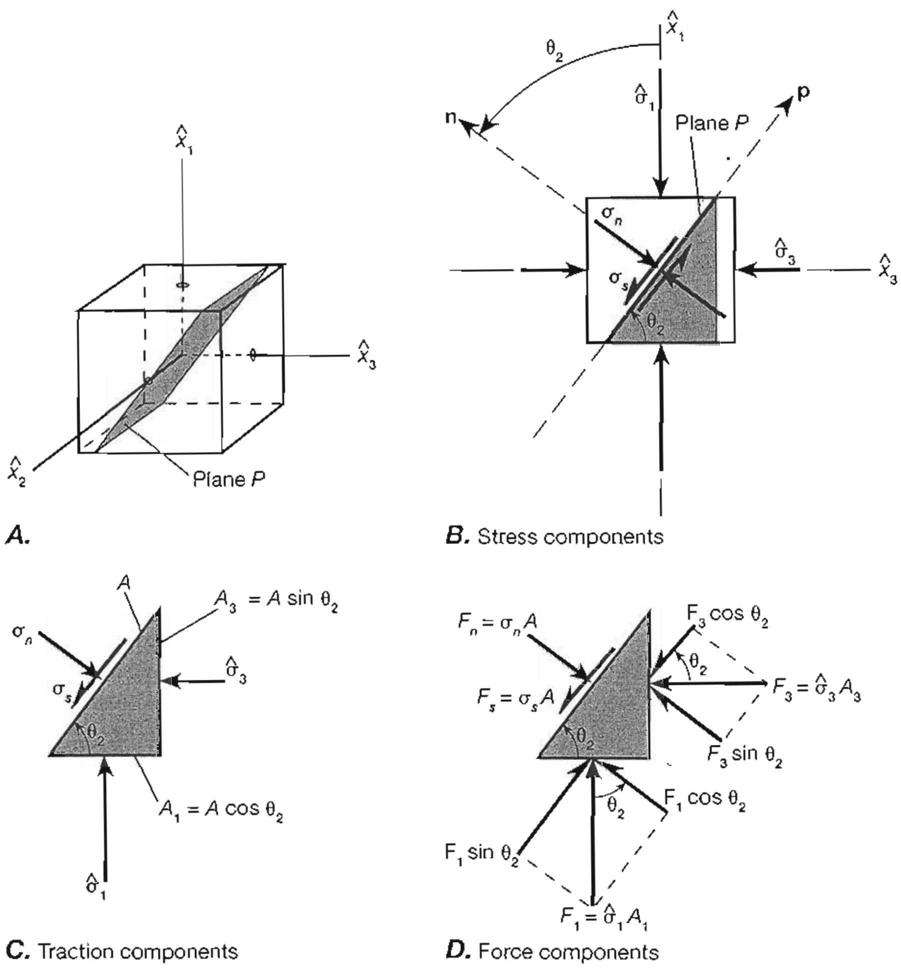


Figure 8.13 Geometry for determining the normal stress and shear stress on a plane *P* of any given orientation through a point. *A*. The plane *P* through the principal coordinate cube is parallel to $\hat{x}_2$ but otherwise of arbitrary orientation. *B*. Two-dimensional view of the geometry in part *A*, showing the distribution of stress components. All stress components and angles are drawn as positive in this diagram. *C*. The triangular element shaded in part *B*, showing only those traction components that act on the exterior of the element. *D*. Diagram of the forces and force components derived from the traction components shown in part *C*.

Convention 1. The general convention for orienting any pair of coordinate axes requires that there is a clockwise sense of rotation from the positive coordinate axis paralleling the greatest normal stress component to the positive coordinate axis paralleling the least normal stress component, regardless of whether, for example, σ_{11} or σ_{33} is the largest. This convention fixes the direction from which we view the diagram and thereby eliminates the ambiguity about whether a shear couple is clockwise or counterclockwise. Thus the principal axes are drawn such that the 90° rotation from positive $\hat{x}_1$ to positive $\hat{x}_3$ is a clockwise rotation.

Convention 2. The orientation of the plane P is defined by the angle θ_2 between the positive $\hat{x}_1$ axis and n, where n is the vector of unit length that is *normal* to P. Positive angles are measured counterclockwise, and we construct the diagram with a positive angle θ_2. The subscript 2 on the angle θ_2 indicates that the angle measures a rotation about the $\hat{x}_2$ axis.

Convention 3. We draw the diagram with positive stress tensor components, according to the geologic sign convention. The normal stress and shear stress components on P are drawn as positive stress tensor components, considering that the vectors n and p (normal and parallel, respectively, to the plane P) are positive coordinate directions that coincide with the positive directions of $\hat{x}_1$ and $\hat{x}_3$ when $\theta_2 = 0$ (Figure 8.13*B*).

Note that with this convention, the positive shear stress component is automatically a counterclockwise shear couple, regardless of the value of θ_2. Thus on any two perpendicular planes for which θ_2 differs by 90°, counterclockwise shear couples are always positive. This result conflicts with the stress tensor sign convention (Figure 8.11*D*, *E*), which dictates that shear couples on perpendicular faces have equal values and opposite shear senses. Thus unavoidably there are different shear stress sign conventions for the Mohr circle and for the stress tensor. The need to shift from one convention to the other when plotting or determining stress tensor components on a Mohr circle is a commou source of error.

We want to determine the normal and shear components (σ_n, σ_s) of the surface stress acting on P. To this end, we isolate in Figure 8.13*C* the shaded triangular element shown in Figure 8.13*B*, and we draw only those traction components that represent the action of the surrounding material on the triangle. Because the infinitesimal square in Figure 8.13*B* is in equilibrium, the triangular element in Figure 8.13*C* must also be in equilibrium, and we can determine the *surface stress components* on P by applying Newton's first law, which requires that the *forces* exerted on the triangular element be balanced.

We convert the traction components into force components by multiplying each traction by the area of the surface on which it acts (Figure 8.13*D*). Although we are dealing with tractions in this derivation, we will persist in using the components and sign convention for the surface stresses, taking care to account for the orientations of the tractions when we add or subtract the forces. In this way, our analysis will give the appropriate value for the surface stress on P. The areas of P and of the sides of the triangular element normal to $\hat{x}_1$ and $\hat{x}_3$ are A, A_1, and A_3, respectively, so the forces acting on the triangular element are

$$F_n = \sigma_n A \quad F_s = \sigma_s A \quad F_1 = \hat{\sigma}_1 A_1 \quad F_3 = \hat{\sigma}_3 A_3 \qquad (8.32)$$

The force on A_1 can be resolved into a pair of components parallel to F_n and to F_s, which are the normal and tangential forces, respecrively, on P (Figure 8.13*D*). The same is true of the force on A_3. Equilibrium of the triangular element is maintained if all forces perpendicular to P sum to zero and if all forces parallel to P sum to zero. From Figure 8.13*D*, these conditions imply that

$$\begin{aligned} F_n - F_1 \cos\theta_2 - F_3 \sin\theta_2 &= 0 \\ F_s - F_1 \sin\theta_2 + F_3 \cos\theta_2 &= 0 \end{aligned} \qquad (8.33)$$

where forces acting in the same direction as F_n or F_s in Figure 8.13*D* are added, and those acting in the opposite direction are subtracted. Rearranging Equation (8.33) to isolate F_n and F_s on the left side, and substituting for the force components from Equation (8.32), we get

$$\begin{aligned} \sigma_n A &= \hat{\sigma}_1 A_1 \cos\theta_2 + \hat{\sigma}_3 A_3 \sin\theta_2 \\ \sigma_s A &= \hat{\sigma}_1 A_1 \sin\theta_2 - \hat{\sigma}_3 A_3 \cos\theta_2 \end{aligned} \qquad (8.34)$$

We can eliminate the area terms from these equations by substituting the following relationships (Figure 8.13*C*)

$$A_1 = A\cos\theta_2 \qquad A_3 = A\sin\theta_2 \qquad (8.35)$$

into Equation (8.34) and dividing through by A. By these manipulations, we express the force balance (Equations 8.33) strictly in terms of the stress components so that they give the results we seek.

$$\begin{aligned} \sigma_n &= \hat{\sigma}_1 \cos^2\theta_2 + \hat{\sigma}_3 \sin^2\theta_2 \\ \sigma_s &= (\hat{\sigma}_1 - \hat{\sigma}_3)\sin\theta_2\cos\theta_2 \end{aligned} \qquad (8.36)$$

Note that all the terms with θ_2 involve products of sine and cosine functions. One of the trigonometric terms comes from resolving the force vectors (Equation 8.33) and the other from resolving the areas (Equation 8.35). The need to resolve both of these quantities to determine stress gives the stress the bi-directional quality that distinguishes it from the unidirectional quality of vectors such as force. Equations (8.12) and (8.13), derived in our numerical example, are the same as Equations (8.36) if in Equations (8.12) and (8.13) we replace σ'_n and σ'_s on the left sides with σ_n and σ_s, respectively, and on the right side we take $\sigma_n = \hat{\sigma}_1$ and realize thar $\hat{\sigma}_3 = 0$.

Thus, given the orientation of any plane defined by θ_2, we can calculate the normal stress and shear stress components on that plane if we know only the principal stresses. These equations, then, justify our earlier assumption that the components of the stress tensor at a point are necessary and sufficient for determining the normal stress and shear stress components on a plane of any orientation through that point.

We can put the equations in a more easily interpreted form by using the standard trigonometric identities:

$$\cos^2\theta_2 = 0.5(1+\cos 2\theta_2) \quad \sin^2\theta_2 = 0.5(1-\cos 2\theta_2) \quad (8.37)$$
$$\sin\theta_2 \cos\theta_2 = 0.5 \sin 2\theta_2$$

Substituting Equations (8.37) into Equations (8.36) and rearranging gives

$$\sigma_n = \left[\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2}\right] + \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right]\cos 2\theta_2 \quad (8.38)$$
$$\sigma_s = \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right]\sin 2\theta_2$$

Here $(\hat{\sigma}_1 + \hat{\sigma}_3)/2$ is the mean normal stress, and $(\hat{\sigma}_1 - \hat{\sigma}_3)/2$ is the maximum possible shear stress, as can be seen from the fact that $\sin 2\theta_2$ in the second equation can be no greater than 1.

Equations (8.38) are identical to Equations (8.28), which we deduced from the geometry of the Mohr circle. Thus Equations (8.38) are the parametric equations for the Mohr circle, with σ_n and σ_s as the variables and θ_2 as the parameter.

We can obtain a more familiar form for the equation of a circle by eliminating θ_2. We rewrite the first Equation (8.38) as

$$\sigma_n - \left[\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2}\right] = \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right]\cos 2\theta_2 \quad (8.39)$$

then square both sides of the second Equation (8.38) and Equation (8.39), and add the resulting two equations together. Applying the trigonometric identity

$$\sin^2 2\theta_2 + \cos^2 2\theta_2 = 1 \quad (8.40)$$

yields the result

$$\left[\sigma_n - \left(\frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2}\right)\right]^2 + \sigma_s^2 = \left[\frac{\hat{\sigma}_1 - \hat{\sigma}_3}{2}\right]^2 \quad (8.41)$$

This equation has the form

$$(x-a)^2 + y^2 = r^2 \quad (8.42)$$

which is the equation of a circle that has its center a distance a along the x axis and has a radius r.

We recommend the following procedure for plotting stress tensor components on the Mohr diagram: Draw a diagram of the coordinate square in physical space, with the coordinate axes oriented relative to each other according to convention (1) above and the stress components appropriately oriented according to the tensor sign convention. Make a table listing the values of the stress tensor components, and then, opposite each component, list its value according to the Mohr diagram sign convention. Normal stress components have the same sign as the tensor components. Determine the sign for the shear stress components by using the diagram of the coordinate square. A shear stress component is positive if it is a counterclockwise couple on the coordinate square, negative if it is a clockwise couple. Finally, plot the values of the components thus determined on the Mohr diagram (see the example given in Appendix 8A).

8.6 Terminology for States of Stress

A number of terms that refer to certain specific states of stress are common in the literature. They all have special characteristics, which are easy to describe in terms of the relevant stress tensor components and Mohr circle diagrams (Figure 8.14).

Hydrostatic pressure, $\hat{\sigma}_1 = \hat{\sigma}_2 = \hat{\sigma}_3 = p$ (Figure 8.14*A*). All principal stresses are compressive and equal. No shear stresses exist on any plane, so all orthogonal coordinate systems are principal coordinates. The Mohr circle reduces to a point on the normal stress axis.

Uniaxial stress. The Mohr diagram for the three dimensional stress is a single circle tangent to the ordinate at the origin. There are two possible cases:

1. *Uniaxial compression,* $\hat{\sigma}_1 > \hat{\sigma}_2 = \hat{\sigma}_3 = 0$ (Figure 8.14*B*). The only stress applied is a compressive stress in one direction. This geometry is commonly used in testing the strength of rock samples in the laboratory.
2. *Uniaxial tension,* $0 = \hat{\sigma}_1 = \hat{\sigma}_2 > \hat{\sigma}_3$ (Figure 8.14C). The only stress applied is a tension in one direction. Engineers often use this geometry to test the mechanical properties of metals.

Axial compression or confined compression, $\hat{\sigma}_1 > \hat{\sigma}_2 = \hat{\sigma}_3 > 0$ (Figure 8.14*D*). A uniaxial compression of magnitude $(\hat{\sigma}_1 - \hat{\sigma}_3)$ is superimposed upon a state of hydrostatic stress $(\hat{\sigma}_2 = \hat{\sigma}_3)$. This state is frequently used in laboratory experiments on the high-temperature, high-pressure properties of rock.

Axial extension, extensional stress, or extension, $\hat{\sigma}_1 = \hat{\sigma}_2 > \hat{\sigma}_3 > 0$ (Figure 8.14*E*). A uniaxial tension of magnitude $(\hat{\sigma}_1 - \hat{\sigma}_3)$ is superimposed on a hydrostatic stress $(\hat{\sigma}_1 = \hat{\sigma}_2)$. This state is also sometimes used in high-temperature, high-pressure laboratory deformation experiments. It is unfortunate that the term *extension* has a different meaning when applied to strain, and the distinction should always be made clear.

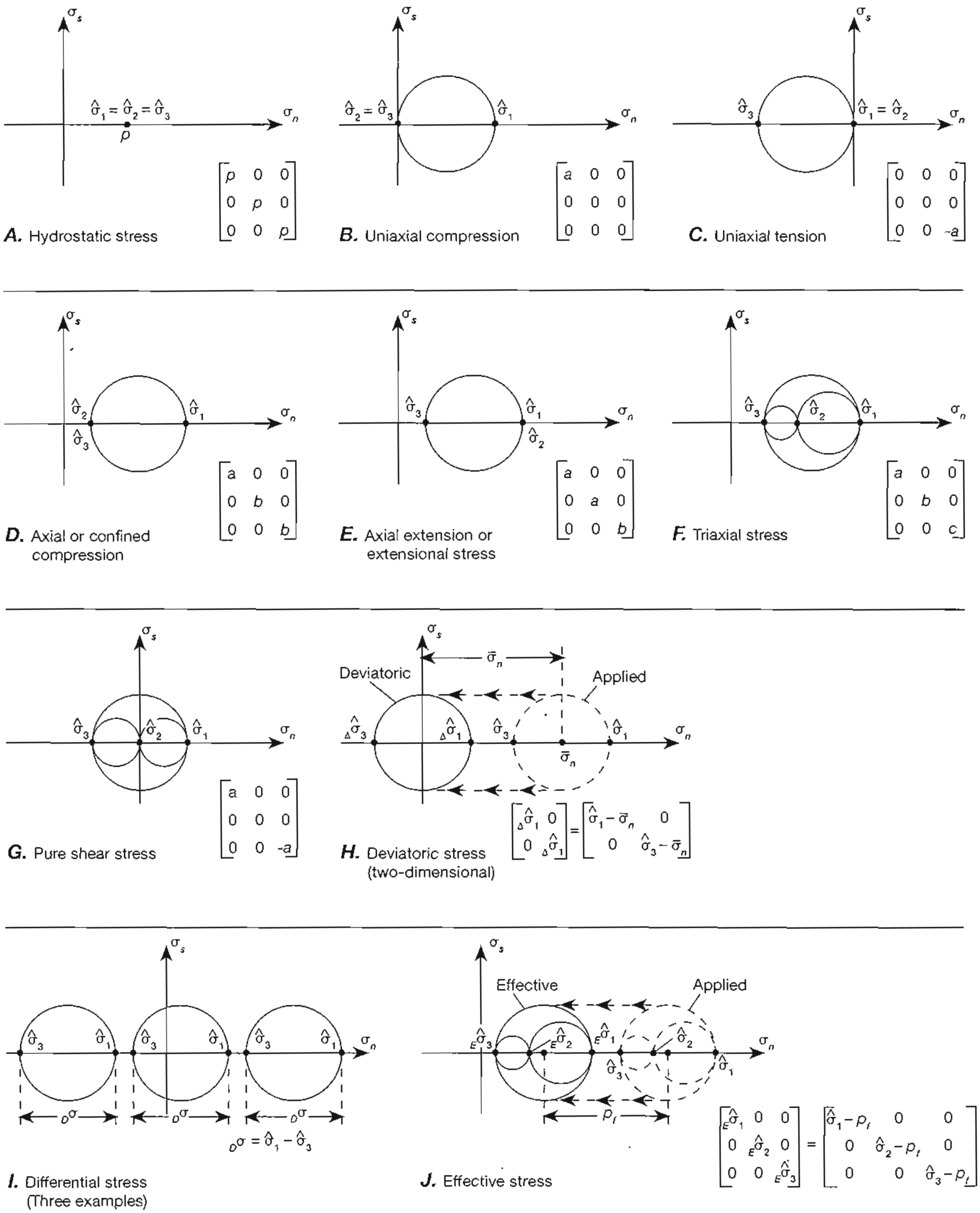


Figure 8.14 Mohr diagrams for special states of stress. The tensor components are shown for the principal coordinate system, with the principal stresses written in standard order from top left to bottom right along the principal diagonal. Here p, a, b, and c all take on positive values, and we assume $a > b > c$.

Triaxial stress, $\hat{\sigma}_1 > \hat{\sigma}_2 > \hat{\sigma}_3$ (Figure 8.14*F*). The principal stresses are all unequal and can be of either sign. The stress plots on the Mohr diagram as three distinct circles (see Box 8.4).

Pure shear stress or pure shear, $\hat{\sigma}_1 = -\hat{\sigma}_3$ and $\hat{\sigma}_2 = 0$ (Figure 8.14*G*). The maximum and minimum principal stresses are equal in magnitude and opposite in sign; the intermediate principal stress is zero. The normal stress on planes of maximum shear stress is zero—hence the name. The Mohr diagram is centered on the origin. The term *pure shear* has a different meaning when applied to strain, and the ambiguity can cause confusion.

Deviatoric stress (Figure 8.14*H*). The components of the deviatoric stress ${}_{\Delta}\sigma_{k\ell}$ are defined by subtracting the mean normal stress $\bar{\sigma}_n$ from each of the normal stress components in the three or two-dimensional stress tensor. In three dimensions,

$${}_{\Delta}\sigma_{k\ell} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \bar{\sigma}_n & 0 & 0 \\ 0 & \bar{\sigma}_n & 0 \\ 0 & 0 & \bar{\sigma}_n \end{bmatrix}$$

$${}_{\Delta}\sigma_{k\ell} = \begin{bmatrix} \sigma_{11} - \bar{\sigma}_n & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \bar{\sigma}_n & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \bar{\sigma}_n \end{bmatrix} \quad (8.43)$$

where

$$\bar{\sigma}_n = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \quad (8.44)$$

In two dimensions in the x_1–x_3 coordinate plane, the components of the deviatoric stress tensor are given by

$${}_{\Delta}\sigma_{k\ell} = \begin{bmatrix} \sigma_{11} - \bar{\sigma}_n & \sigma_{13} \\ \sigma_{31} & \sigma_{33} - \bar{\sigma}_n \end{bmatrix} \quad (8.45)$$

where

$$\bar{\sigma}_n = \frac{\sigma_{11} + \sigma_{33}}{2} \quad (8.46)$$

For the deviatoric stress in two dimensions, the center of the Mohr eircle is shifted to the origin of the graph so that it appears to be a pure shear stress (Figure 8.13*H*). The deviatoric stress is useful in describing the behaviour of a material that depends only on the size of the Mohr circle, which is a measure of the maximum shear stress, and not on the location of the Mohr circle along the normal stress axis, which is a measure of the average pressure.

Differential stress (Figure 8.14*I*). The differential stress ${}_{D}\sigma$ is the difference between the maximum and minimum principal stresses:

$${}_{D}\sigma = \hat{\sigma}_1 - \hat{\sigma}_3 \quad (8.47)$$

It is always a positive scalar quantity that is twice the radius of the largest Mohr circle and therefore twice the maximum shear stress. For a two dimensional stress, it is the diameter of the Moht circle ($2r$; see the second Equation 8.26) and is therefore a scalar invariant of the stress tensor (Section 8.3, ptoperty 5). For a state of axial compression or axial extension (Figure 8.14*D, E*), it is the uniaxial stress that is applied in addition ro the hydrostatic stress.

Effective stress (Figure 8.14*J*). The components of the effective stress tensor ${}_{E}\sigma_{kl}$ are defined in three dimensions by

$${}_{E}\sigma_{k\ell} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} p_f & 0 & 0 \\ 0 & p_f & 0 \\ 0 & 0 & p_f \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - p_f & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p_f & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p_f \end{bmatrix} \quad (8.48)$$

where $\sigma_{k\ell}$ are the components of the applied stress, and p_f is the pressure of the pore fluid in the rock. As shown in the diagram, the effective stress is the result of a shift of the Mohr circle toward lower normal stresses by an amount equal to rhe pote fluid pressure p_f. We discuss the effective stress in greater detail in Section 9.5, where we show that the mechaniccal behavior of a brirrle material depends on the effective stress, not on the applied stress.

Additional Readings

Eringen, A. C. 1967. *Mechanics of continua*. New York: Wiley.

Fung, Y. C. 1965. *Foundations of solid mechanics*. Englewood Cliffs, N. J.: Prentice-Hall.

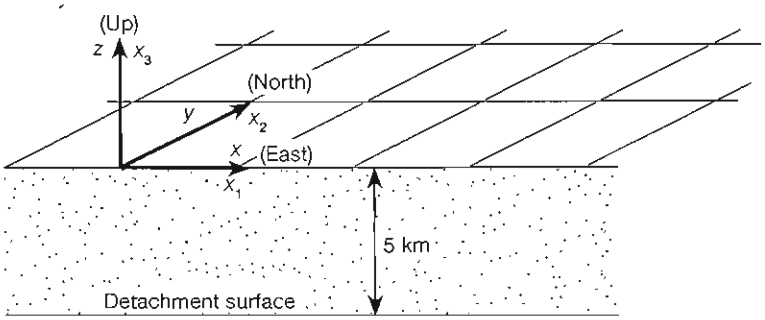
Hubbert, M. K. 1972. *Structural geology*. Hafner Publishing Co.

Means, W. D. 1976. *Basic concepts of stress and strain for geologists*. New York: Springer-Verlag.

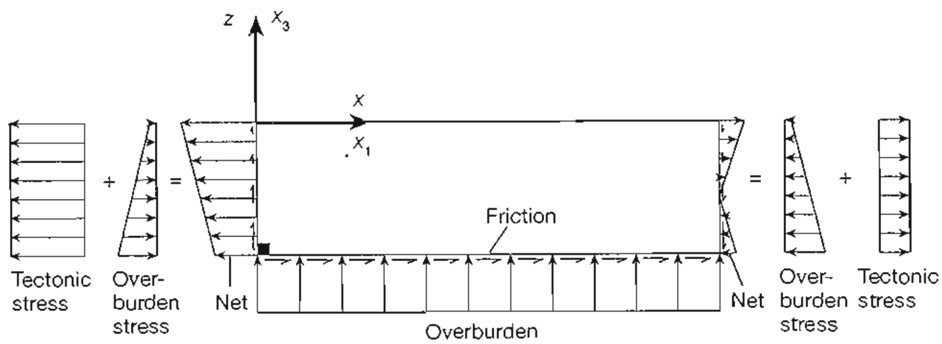
Appendix 8A: An Illustrative Problem

In this appendix, we discuss a specific numerical problem to illustrate the technique of using the Mohr circle and to illustrate the types of problems that one can solve using a Mohr circle. Students who have not read Section 8.4 should ignore, for each question, the discussions about the change ftom tensor to Moht diagram sign convention and should simply start with the values of the stress components given for the Mohr diagram sign convention.

Consider a fault block that is 5 km thick and tests on a horizontal detachment associated with a listric normal fault. The coordinate system is shown in Figure 8A.1*A*. Figure 8A.1*B* shows a free body diagram of a section of the detachment sheet. The action of the material that originally surtounded the free body is indicated by a distribution of traction components. These tractions arise from the force of gravity (the overburden), the applied tectonic stress (which we assume to be an east–west horizontal tensile stress), and the frictional resistance to sliding on the detachment. We determine the stress at the bottom left corner of the free body, where we know the tractions acting on both co-



A.



B.

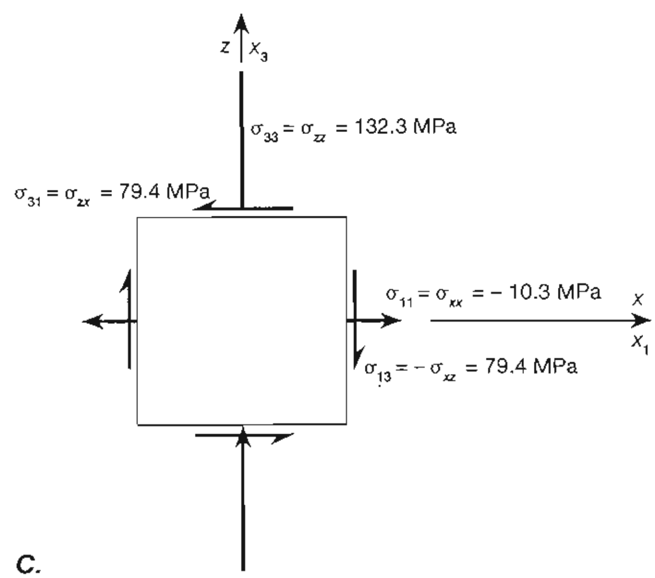


Figure 8A.1 Simplified model of a fault block resting on a detachment. *A.* A fault block bounded below by a horizonral derachmenr fault. The coordinate axes are used in the analysis. *B.* A free-body diagram of an isolared parr of the fault block. *C.* The coordinate square from rhe lower-left corner of rhe isolated body of the faulr block.

C.

ordinate planes. The corner is shown enlarged in Figure 8A.1C.

The vertical normal stress σ_{33} is the overburden pressure due to gravity, and it equals the weight per unit area of the overlying rock. This is

$$\sigma_{33} = \rho g h \tag{8A.1}$$

where ρ is the mass density of the rock, g is the gravitational acceleration, and h is the distance to the top surface. The stress is compressive and therefore positive.

The horizontal normal stress σ_{11} is the sum of the horizontal compressive stress due to the overburden and the tectonically applied stress T. Because we assume the rock has some finite strength, the part due to the overburden is some fraction $\kappa < 1$ of the vertical normal stress (in a fluid, however, $\kappa = 1$). We assume, furthermore, that T is a tensile tectonic stress, constant with depth, that pulls the fault block to the west. Thus

$$\sigma_{11} = \kappa(\rho g h) - T \tag{8A.2}$$

where we subtract T because it is tensile. σ_{11} could be positive or negative, depending on the relative values of the overburden and of T.

The horizontal normal stress σ_{22} results only from the overburden, because there is no externally applied stress parallel to the x_2 axis.

$$\sigma_{22} = \kappa(\rho g h) \tag{8A.3}$$

Assuming that no shear stresses act on the plane of the diagram, we have

$$\sigma_{21} = \sigma_{23} = 0 \tag{8A.4}$$

The frictional shear stress along the detachment σ_{31} is given by the product of the coefficient of friction μ and the normal stress across the sliding surface. Applying the geologic tensor sign convention, we see that on the negative side of the coordinate surface, the frictional resistance acts in a positive coordinate direction (x_1). Thus this shear stress component must be positive.

$$\sigma_{31} = \mu(\rho g h) \tag{8A.5}$$

Because of the symmetry of the stress tensor (Equations 8.24), we have now determined all the independent components of the stress tensor in the given coordinate system.

In order to introduce definite numbers into the analysis, we adopt the following geologically reasonable values for the symbols in the equations:

$$\rho = 2700 \text{ kg/m}^3 \qquad h = 5000 \text{ m}$$

$$g = 9.8 \text{ m/s}^2 \qquad T = 50 \text{ MPa}$$

$$\kappa = 0.3$$

$$\mu = 0.6$$

Using these values with Equations (8A.1) to (8A.5), and using the symmetry condition for the stress tensor (Equations 8.24), we obtain the following values in megapascals for the components of the stress tensor:

$$\sigma_{k\ell} = \begin{bmatrix} -10.3 & 0 & 79.4 \\ 0 & 39.7 & 0 \\ 79.4 & 0 & 132.3 \end{bmatrix} \tag{8A.6}$$

Because $\sigma_{21} = \sigma_{23} = 0$, the plane normal to x_2 must be a principal plane, x_2 must be a principal axis, and σ_{22} must be a principal stress. Without knowing the values of the other principal stresses, however, we do not know whether it is the maximum, the intermediate, or the minimum principal stress. We do know that the other two principal axes must lie in the x_1–x_3 plane. On the basis of the discussion at the end of Section 8.2 and that at the beginning of Section 8.5, we conclude that because x_2 is a principal axis, we can analyze the stresses on planes parallel to x_2 by using a two-dimensional analysis of stress components in the x_1–x_3 plane.

In the following discussion, we refer by number to the properties of the Mohr circle that we discussed in Sections 8.3 and Box 8.4 and to the conventions listed in Section 8.5, and we do not duplicate those discussions here.

Question 1

Construct the Mohr circle for the two-dimensional stress acting on planes normal to the x_1–x_3 plane—that is, parallel to x_2.

Procedure

To obtain the components of the two-dimensional stress tensor in the x_1–x_3 plane, we drop all components that have a 2 in the subscripts (see the discussion at the beginning of Section 8.5), which eliminates the second row and the second column of the matrix in Equation (8A.6), leaving, in megapascals,

$$\sigma_{k\ell} = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} -10.3 & 79.4 \\ 79.4 & 132.3 \end{bmatrix} \tag{8A.7}$$

Next we must determine how these components plot on the Mohr diagram. The steps involved are as follows: (1) Draw a diagram of the stress components in physical space. (2) Use this diagram to change the stress components from the tensor sign convention (Section 8.4 and Figure 8.11) to the Mohr circle sign convention (Section 8.2 and Figure 8.3). (3) Plot the stress components on the Mohr diagram. (4) Coustruct the Mohr circle.

Discussion

1. Before constructing a diagram of the stress components, we must be sure the coordinates in Figure 8A.1C are drawn in accordance with convention 1 (see Section 8.5). If σ_{11} had been greater than σ_{33},

then we would have had to plot x_1 positive to the left and x_3 positive up, which is equivalent to viewing the diagram from the north instead of from the south as shown in Figure 8A.1*A*.

On the coordinate square (Figure 8A.1C), draw the pairs of arrows to represent the stress components, using the tensor sign convention and the values of the stress components (Equation 8A.7) to determine the correct orientations.

2. From Figure 8A.1C, we can determine the signs of the stress components appropriate for plotting on the Mohr diagram. We recommend constructing a table such as this:

Tensor value	Mode	Mohr diagram value
$\sigma_{11} = -10.3$	tensile	$\sigma_{xx} = -10.3$
$\sigma_{13} = 79.4$	clockwise	$\sigma_{xz} = -79.4$
$\sigma_{33} = 132.3$	compressive	$\sigma_{zz} = 132.3$
$\sigma_{31} = 79.4$	counterclockwise	$\sigma_{zx} = 79.4$

In the first column, list the symbols and values for the stress tensor components exactly as they are given in Equation (8A.7). Use Figure 8A.1C to check whether the normal stress components are tensile or compressive and whether the shear stress components are clockwise or counterclockwise; note this in the second column. This second column, with the sign conventions for the Mohr circle given in Figure 8.3, determines the sign of each stress component entered in the third column.

3. The pairs of values $(\sigma_{xx}, \sigma_{xz})$ and $(\sigma_{zz}, \sigma_{zx})$ from the last column plot as two points on the Mohr diagram (Figure 8A.2*A*).
4. A line connecting these two points on the Mohr diagram must be a diameter of the Mohr circle (property 3v), and the point where the diameter intersects the σ_n-axis is the center of the circle (property 5i). The circle can then be drafted with a drafting compass. Alternatively, we can use equations of the form of (8.27) to calculate the center and radius of the Mohr circle, from which the whole circle can be constructed.

$$\bar{\sigma}_n = \frac{\sigma_{xx} + \sigma_{zz}}{2} = \frac{-10.3 + 132.3}{2} = 61 \text{ MPa} \qquad (8A.8)$$

$$r = 0.5[(\sigma_{xx} - \sigma_{zz})^2 + (2\sigma_{xz})^2]^{0.5}$$

$$= 0.5[(-10.3 - 132.3)^2 + 4(79.4)^2]^{0.5}$$

$$r = 106.7 \text{ MPa} \qquad (8A.9)$$

Question 2

What are the values and orientations of the principal stresses in the x_1–x_3 (or x–z) plane? Draw a diagram of physical space showing the relationship between the $x_1 - x_3$ $(x - z)$ coordinates and the principal coordinates in that plane.

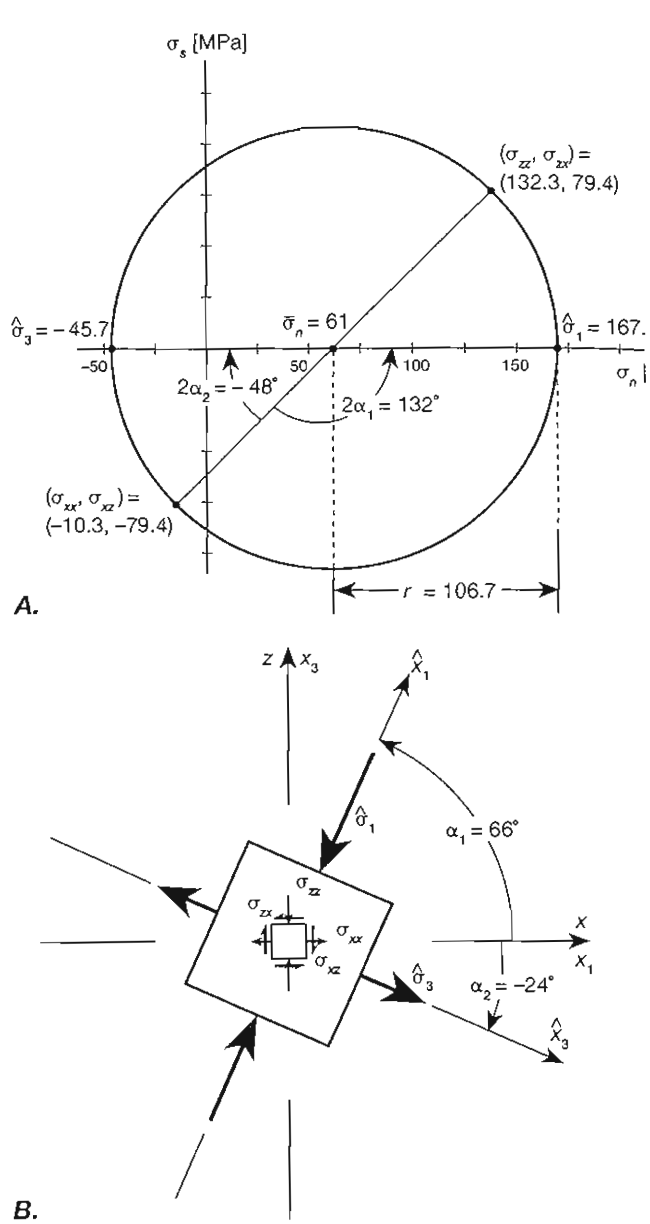


Figure 8A.2 Mohr circle construction for the illustrative problem. *A*. Mohr circle for the stress under consideration. *B*. Orientation of the principal axes in physical space and their relationship to the original coordinate system, as derived from the Mohr circle in part *A*. The two differently oriented coordinate squares both represent the infinitesimal point, and the components on each square represent the same stress. The squares are drawn different sizes for clarity; arrows are not to scale.

Procedure

The values of the principal stresses are read from Figure 8A.2*A* at the points where the Mohr circle intersects the σ_n-axis (property 2i). The values are 167.7 MPa and

-45.7 MPa. These values can also be obtained by adding and subtracting the magnitude of the radius of the Mohr circle (Equation 8A.9) to the value of $\overline{\sigma}_n$, the center of the circle (Equation 8A.8). Recalling that the third principal stress is 39.7 MPa (Equation 8A.6), we label the values in decreasing order such that the stress in principal coordinates is given by

$$\hat{\sigma}_{k\ell} = \begin{bmatrix} \hat{\sigma}_1 & 0 & 0 \\ 0 & \hat{\sigma}_2 & 0 \\ 0 & 0 & \hat{\sigma}_3 \end{bmatrix} = \begin{bmatrix} 167.7 & 0 & 0 \\ 0 & 39.7 & 0 \\ 0 & 0 & -45.7 \end{bmatrix} \quad (8A.10)$$

The orientations of the principal axes are also determined from Figure 8A.2*A*, using the properties 3i and ii and the plotting conventions used for Figure 8.13*B*. We recommend tabulating the measurements as shown below to avoid confusion and to ensure proper observance of the conventions. Measurements on the Mohr diagram are shown in Figure 8A.2*A*, and the corresponding measurements in physical space are shown in Figure 8A.2*B*.

On the Mohr Diagram

Angle	Sense of angle	Measured from	Measured to
$2\alpha_1 = 132°$	counter-clockwise	$(\sigma_{xx}, \sigma_{xz}) =$ $(-10.3, -79.4)$	$(\hat{\sigma}_1, 0) =$ $(167.7, 0)$
$2\alpha_2 = -48°$	clockwise	$(\sigma_{xx}, \sigma_{xz}) =$ $(-10.3, -79.4)$	$(\hat{\sigma}_3, 0) =$ $(-45.7, 0)$

In Physical Space

Angle	Sense of angle	Measured from	Measured to
$\alpha_1 = 66°$	counter-clockwise	x (x_1)	$\hat{x}_1$
$\alpha_2 = -24°$	clockwise	x (x_1)	$\hat{x}_3$

Discussion

On the Mohr diagram, the angle $2\alpha_1 = 132°$ is measured from the radius at $(\sigma_{xx}, \sigma_{xz})$ to the radius at $(\hat{\sigma}_1, 0)$. It is twice the angle $\alpha_1 = 66°$ in physical space measured from the coordinate axes x to $\hat{x}_1$, which are the respective normals to the planes on which the stress components act. We measure angles in physical space from x, because it is a coordinate axis whose orientation is known. The angles on the Mohr circle must therefore be measured from the point representing the stresses on the plane normal to x (property 3i). We could use the z-axis as a reference in the same way, in which case the corresponding angles on the Mohr circle would be measured from the radius at $(\sigma_{zz}, \sigma_{zx}) = (132.3, 79.4)$, and the corresponding angles in physical space would be measured from the z-axis.

In labeling the principal axes, we form a right-handed coordinate system (see Figure 8.1.2 in Box 8.1) with $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ parallel respectively to the maximum, intermediate, and minimum compressive stresses and with the positive ends of the principal axes arranged such that they conform to convention 1, discussed in Section 8.5 (Figure 8A.2*B*). Here $\hat{x}_1$ and $\hat{x}_3$ are necessarily perpendicular, because they are the normals to planes whose stress components plot at opposite ends of a diameter of the Mohr circle (Figure 8A.2*A*) (properties 3v).

Question 3

What are the extreme absolute values of the shear stress acting on planes normal to the x_1–x_3 (x–z) plane—that is, parallel to x_2 (or y)—and what are the orientations of the planes on which these values occur?

Procedure

The maximum absolute values of the shear stress are read directly from the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle (property 4) (Figure 8A.3*A*).

$$|\sigma_s|_{(\max)} = 106.7 \text{ MPa}$$

The orientations of the normals to the planes of maximum shear stress are determined from the angles on the Mohr circle (Figure 8A.3*A*) as tabulated below:

On the Mohr Diagram

Angle	Sense of angle	Measured from	Measured to
$2\theta = +90°$	counter-clockwise	$(\hat{\sigma}_1, 0) =$ $(167.7, 0)$	(61, 106.7)
$2\theta = -90°$	clockwise	$(\hat{\sigma}_1, 0) =$ $(167.7, 0)$	(61, -106.7)

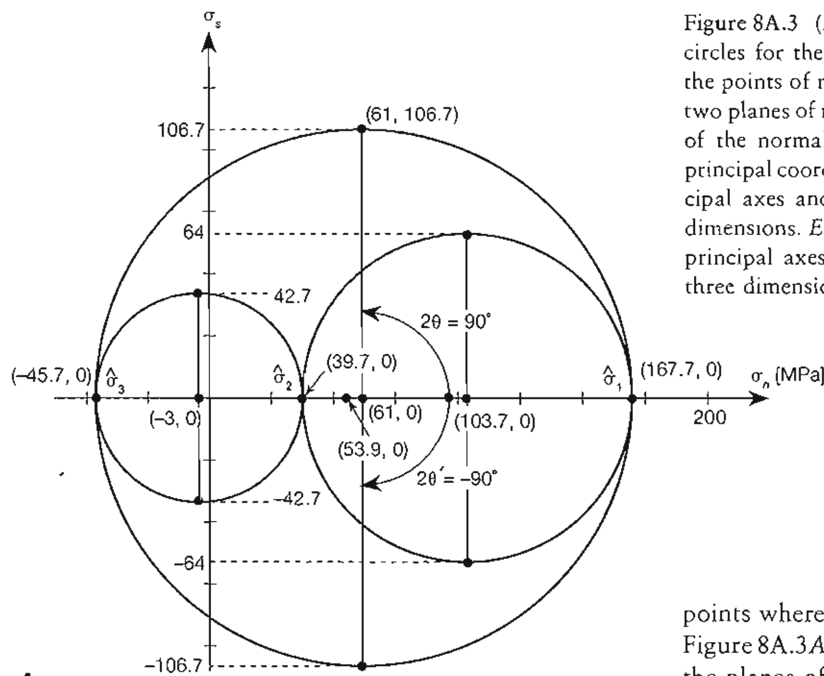
In Physical Space

Angle	Sense of angle	Measured from	Measured to
$\theta = +45°$	counter-clockwise	$\hat{x}_1$	n^+
$\theta' = -45°$	clockwise	$\hat{x}_1$	n^-

where n^+ and n^- are, respectively, the normals to the planes P^+ and P^- on which the maximum shear stress is respectively positive and negative in the Mohr circle sign convention (Figure 8A.3*B*, *C*).

Discussion

The value of the maximum shear stress is simply given by the length of the radius of the appropriate Mohr circle. In the $\hat{x}_1$–$\hat{x}_3$ plane, which is also the x–z plane, $|\sigma_s|$ is a maximum on the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle at those



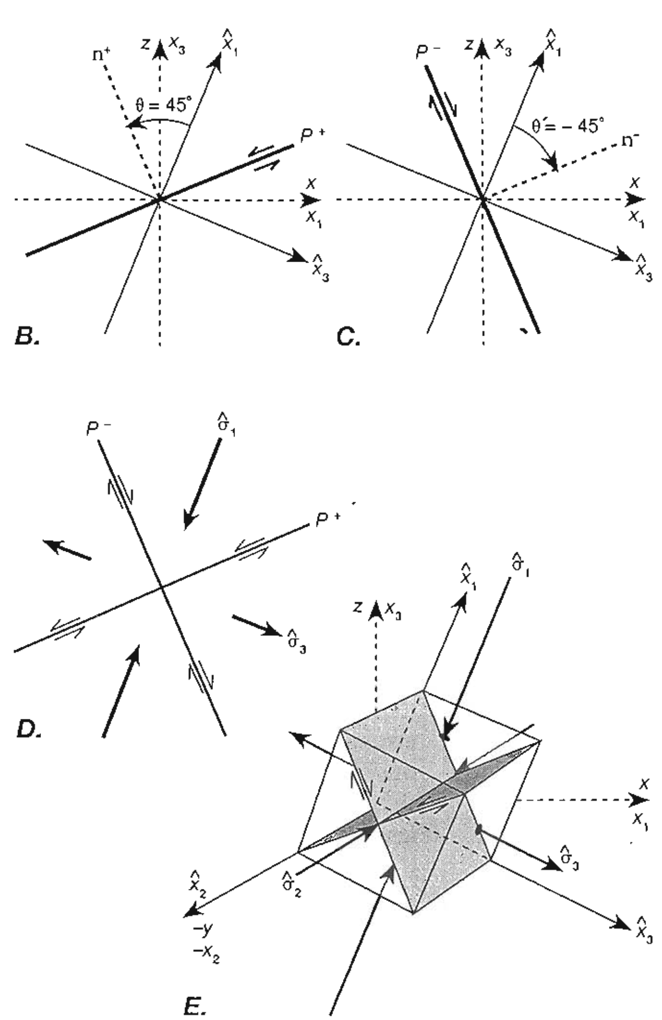


Figure 8A.3 (*Left*) Planes of maximum shear stress. *A*. Mohr circles for the three-dimensional stress, with radii drawn to the points of maximum shear stress on each circle. *B*, *C*. The two planes of maximum shear stress, showing the orienrations of the normals to the planes relative to the reference and principal coordinate axes. *D*. Relative orientations of the principal axes and the planes of maximum shear stress in two dimensions. *E*. Relative orienration of the reference axes, the principal axes, and the planes of maximum shear stress in three dimensions.

points where the radius is normal to the σ_n-axis. From Figure 8A.3*A*, $2\theta = -2\theta' = 90°$. Thus in physical space, the planes of maximum shear stress are parallel to $\hat{x}_2$, and their normals are at $\theta = +45°$ (Figure 8A.3*B*) and $\theta' = -45°$ (Figure 8A.3C) from $\hat{x}_1$. The planes of maximum shear stress are therefore perpendicular to each other. They are called the conjugate planes of maximum shear stress, and the relationship between these planes and the principal stresses is shown in Figure 8A.3*D*.

Question 4

In the three-dimensional solid, what are the absolute values of the maximum shear stress and the orientations of the planes on which these values occur?

Procedure

In order to answer this question, we must consider the Mohr diagram for the three-dimensional stress (Box 8.4). When all three Mohr circles are plotted (Box 8.4, properties 1iii and 4ii; Figure 8A.3*A*), it is clear that the maximum absolute value of the shear stress occurs on the largest Mohr circle, which is the $\hat{\sigma}_1$–$\hat{\sigma}_3$ circle (Box 8.4, property 4ii). The planes of maximum shear stress in three-dimensional physical space are shown in Figure 8A.3*E*. The maximum shear stress on the $\hat{\sigma}_1$–$\hat{\sigma}_2$ Mohr circle (64 MPa) and on the $\hat{\sigma}_2$–$\hat{\sigma}_3$ Mohr circle (42.7 MPa) are maxima only for the respective sets of planes that are parallel to $\hat{x}_3$ and $\hat{x}_1$.

Question 5

What is the mean normal stress for the two-dimensional case in the $\hat{x}_1$–$\hat{x}_3$ plane? What is the mean normal stress for the three-dimensional case?

Procedure

The two-dimensional mean normal stress for the set of planes parallel to any of the principal axes can be determined from the Mohr diagram: Simply read off the value of rhe normal stress at the center of the appropriate Mohr circle (Figure 8A.3*A*). Alternatively, it can be calculated using an equation of the form of Equation (8.26). For the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle, we have

$$\overline{\sigma}_n = \frac{\hat{\sigma}_1 + \hat{\sigma}_3}{2} = \frac{167.7 - 45.7}{2} = 61 \text{ MPa}$$

The three-dimensional mean normal stress cannor be read off the Mohr diagram in any simple way (Box 8.4, property 5ii). It must be calculated from Equation (8.4.4).

$$\overline{\sigma}_n = \frac{\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3}{3} = \frac{167.7 + 39.7 - 45.7}{3} = 53.9 \text{ MPa}$$

Discussion

In the two-dimensional case, $\overline{\sigma}_n$ is the center of the appropriate Mohr circle, and it is the value of rhe normal stress on the planes of maximum shear stress that parallel one of the principal axes. In the foregoing example, we considered the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle, which shows the stresses on planes parallel to $\hat{x}_2$. In the three-dimensional case, the mean normal stress $\overline{\sigma}_n$ has neither of these properties (Figure 8A.3*A*).

Question 6

What are the values of the normal and shear components of the surface stress that acts on each of the following planes?

Plane A is parallel to $\hat{x}_2$, and its normal is at an angle $\alpha_A = 35°$ from x_1 (Figure 8A.4*A*, *B*).

Plane B is parallel to $\hat{x}_1$, and its normal is at an angle $\theta_B = -30°$ from $\hat{x}_3$ (Figure 8A.4*C*, *D*).

Procedure

Because plane A is parallel to $\hat{x}_2$, which is a principal axis, the normal $\mathbf{n}_A$ to the plane lies in the $\hat{x}_1$–$\hat{x}_3$ plane, which is also the x_1–x_3 (or x–z) plane (Figure 8A.4*A*, *B*). The stress componenrs on this plane must therefore plot on the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle (Figure 8A.4*E*). Similarly, plane B is parallel to the principal axis $\hat{x}_1$, so its normal $\mathbf{n}_B$ lies in the $\hat{x}_2$–$\hat{x}_3$ plane (Figure 8A.4*C*, *D*). The srress componenrs on plane B, therefore, must plot on the $\hat{\sigma}_2$–$\hat{\sigma}_3$ Mohr circle (Figure 8A.4*E*). This geometry makes it possible to solve both problems by separate two-dimensional analyses (see the discussion at the beginning of Section 8.5). The geometric relationships in the appropriate two-dimensional planes are shown for planes A and B in Figures 8A.4*B*, *D*, respectively, where the conventions for plotting coordinate axes (convention (1), Section 8.5) have been used.

The relationships are summarized below in the table "In Physical Space" (Figure 8A.4*B*, *D*). The construction of the Mohr diagram that defines the stress components on the relevant planes is derived from properties 3i and ii and from the data in this table. It is summarized below in the table "On the Mohr Circle" (Figure 8A.4*E*).

In physical space (Figure 8A.4*B*), the normal $\mathbf{n}_A$ to plane A is defined by the angle $\alpha_A = 35°$ measured counterclockwise from x (x_1) in the x–z (x_1–x_3) plane, which is the same as the $\hat{x}_1$–$\hat{x}_3$ plane. Here x is the normal to the plane on which the stress components are (σ_{xx}, σ_{xz}) = (−10.3, −79.4). Thus we can find rhe stress components on plane A on the $\hat{\sigma}_1$–$\hat{\sigma}_3$ Mohr circle by measuring an angle $2\alpha_A = 70°$ eounterclockwise from

In Physical Space

Angle	Coordinate plane containing rhe angle	Sense of angle	Measured from	Measured to
Plane A $\alpha_A = 35°$	$\hat{x}_1$–$\hat{x}_3$ and x_1–x_3 (or x–z)	counterclockwise	x_1 (or x)	$\mathbf{n}_A$
Plane B $\theta_B = 30°$	$\hat{x}_2$–$\hat{x}_3$	clockwise	$\hat{x}_3$	$\mathbf{n}_B$

On The Mohr Diagram

Mohr circle	Angle	Sense of angle	Measured from	Measured to
Plane A $\hat{\sigma}_1$–$\hat{\sigma}_3$	$2\alpha_A = 70°$	counterclockwise	(−10.3, −79.4)	(σ_n, σ_s) on plane A
Plane B $\hat{\sigma}_2$–$\hat{\sigma}_3$	$2\theta_B = -60°$	clockwise	(−45.7, 0)	(σ_n, σ_s) on plane B

the radius at the point $(\sigma_{xx}, \sigma_{xz}) = (-10.3, -79.4)$ (Figure 8A.4*E*). A similar procedure is used to find the stress components on plane B (Figure 8A.4*D*), except that in this case we must use the $\hat{\sigma}_2$–$\hat{\sigma}_3$ Mohr circle, and angles are measured from $\hat{x}_3$ in physical space and from $(\hat{\sigma}_3, 0) = (-45.7, 0)$ on the Mohr diagram.

The normal stress and shear stress components on planes A and B are read from the appropriate Mohr circles in Figure 8A.5*E*. The results, using the Mohr circle sign conventions for the stress components, are

For plane A: $(\sigma_n, \sigma_s) = (111.1, -94.2)$

For plane B: $(\sigma_n, \sigma_s) = (-24.4, 37.0)$

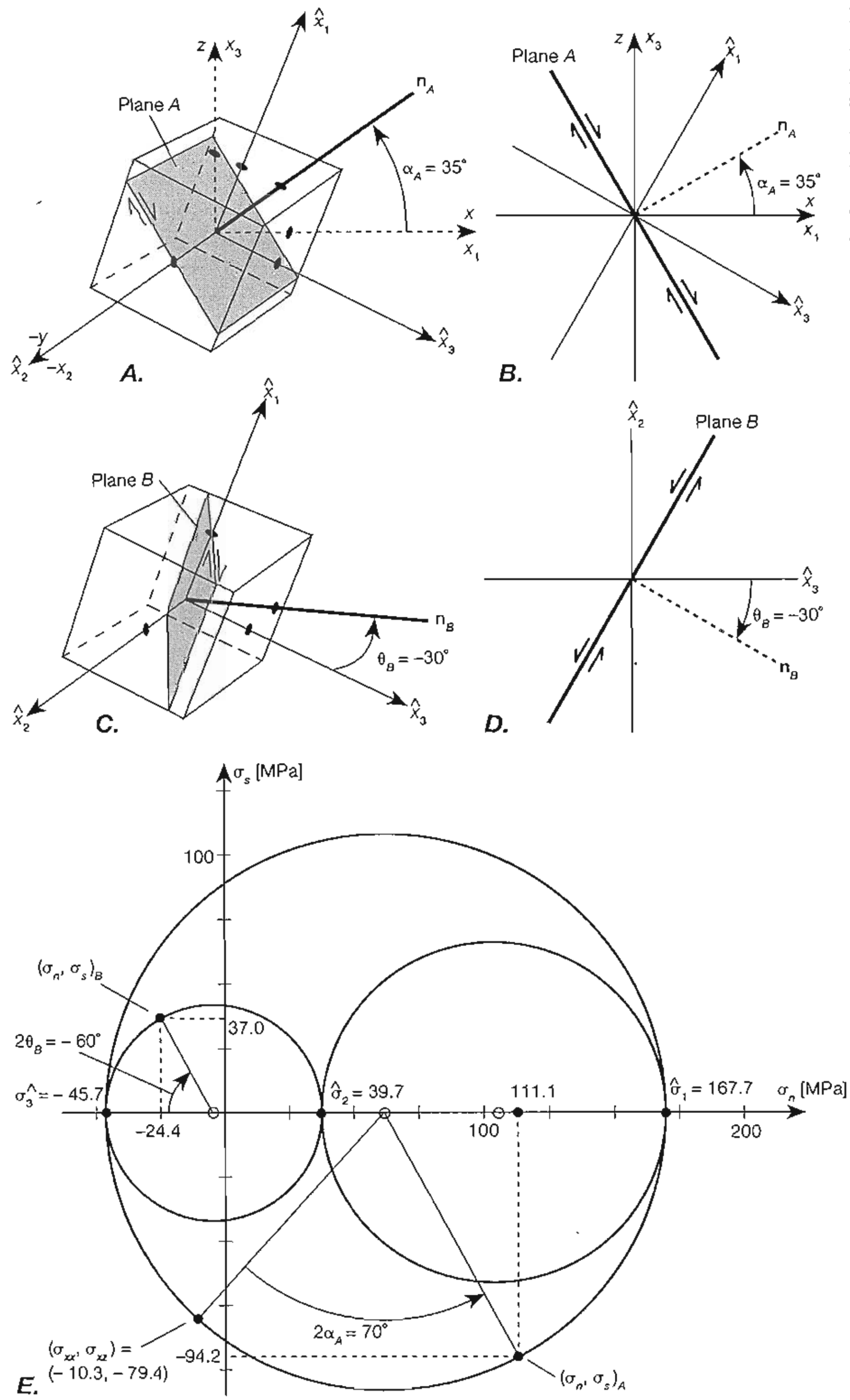


Figure 8A.4 Determining the components of surface stress on plane *A* and plane *B*. *A*. Three-dimensional diagram showing the orientation of plane *A* with respect to the principal coordinate planes. *B*. Two-dimensional diagram of the relationships in part *A*. *C*. Three-dimensional diagram showing the orientation of plane *B* with respect to the principal coordinate planes. *D*. Two-dimensional diagram of the relationships in part C. Note how the principal axes have been oriented relative to one another to conform to the convention shown in Figure 8.4.1C in Box 8.4. *E*. Mohr diagram of three-dimensional stress, showing the construction for determining the components of surface stress on planes *A* and *B*.