CHAPTER

15 Geometry of Homogeneous Strain

To further our understanding of the origin and significance of the folds, foliations, and lineations discussed in the lasr four chapters, we need ro become more familiar with the nature of strain, as manifested in rocks. We introduced some concepts of strain in Chapters 7, 9, 12, and 14, but we need a more thorough and systematic understanding in order to evaluate theoretically the models proposed for formation of ductile structures, as well as to test these models against observations of natural deformation.

Our approach is largely geometric and qualitative, because our intent is to provide intuition into the physical characteristics of deformation, and strain lends itself easily to geometric description. The quantitative analysis of the ideas discussed in this chapter requires a rigorous mathematical treatment of strain, which we introduce in Box 15.1, and which is developed in depth in more advanced books on continuum mechanics and its geologic applications (see the list of readings at the end of this chapter). Readers interested in this approach should read through Section 15.2 before reading Box 15.1.

The strain of a body is simply the change in *size* and *shape* that the body has experienced during deformation. The strain is homogeneous if the changes in size and shape are proportionately identical for each small part of the body and for the body as a whole (Figure 15.1*A*, *B*). A consequence of these conditions is that for any homogeneous strain, planar surfaces remain planar, straight lines remain straight and parallel planes and lines remain parallel. The srrain is inhomogeneous (Figure 15.1*A*, *C*) if the changes in size and shape of small parts of the body are proportionately different from place to place and different from rhat of the body as a whole. Straight lines become curved, planes become curved surfaces, and parallel planes and lines generally do not remain parallel after deformation.

The strain must be inhomogeneous during folding, because in such a deformation, planes and lines do not generally remain planar, straight, or parallel. Within very small volume-elements, however, the strain is statistically homogeneous, and we describe an inhomogeneous strain as a variation of homogeneous strain from place to place in the structure. We discuss how big such a "small" volume-element musr be in Section 15.7.

The progressive deformation of a body refers to the motion that carries the body from its initial undeformed state to its final deformed state. The strain states through which the body passes during a progressive deformation define the strain path. The state of strain of a body is the net result of all the deformations the body has undergone. Although all states of strain are the result of progressive deformation, the final state of strain provides no information about the particular strain path that the body experienced.

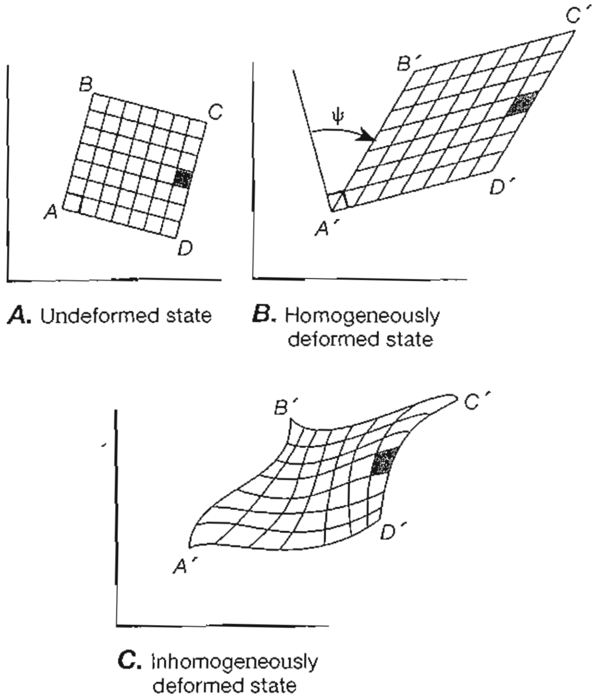


Figure 15.1 Homogeneous and inhomogeneous plane deformation of a material square. *A* and *B*. Homogeneous strain. The small black square is strained in exactly the same way as the whole square and as all the other squares. ψ is the angle of shear. *A* and *C*. Inhomogeneous strain. The small black square is sufficiently small that its strain is essentially homogeneous, but it is not identical to the strain of the whole square or to that of any of the other small squares.

Strain in general must be described in three dimensions, because the size and shape of a body are three-dimensional characteristics. In much of our discussion, however, we consider only a two-dimensional deformation called plane strain, in which the strain is completely described by changes in size and shape in a single orientation of plane through the body, and no deformation occurs normal to that plane. Although plane strain is commonly used to analyze deformation, its application to many situations in natural rock deformation is, strictly speaking, unjustified. Nevertheless, the geometry of two-dimensional deformation is intuitively easier to understand, and the generalization to three dimensions adds considerable complexity but little insight into the geometric characteristics of deformation. For these reasons we concentrate on the properties of two-dimensional strain.

In discussing the geometry of strain, we refer to geometric objects such as lines, planes, circles, and ellipses. Such geometric objects are called material objects if they are always defined by the same set of material particles. A bedding plane, for example, is a material plane because no matter how it moves and deforms, it is always defined by the same set of material particles. A coordinate plane defined by two reference axes, on the other hand, is a nonmaterial plane because as a body deforms, its material particles can move through the coordinate plane and, consequently, different sets of material particles occupy the coordinate plane at different times. This distinction is important in the subsequent discussion.

15.1 Measures of Strain

Linear Strain

The size of a body is measured by its volume, which in turn is proportional to the product of three characteristic lengths of the body. For example, the volume V of a rectangular block that has edges of lengths ℓ_1, ℓ_2, and ℓ_3 is $V = \ell_1 \ell_2 \ell_3$, and the volume of an ellipsoid that has semiaxes of lengths r_1, r_2, and r_3 is $V = [4/3]\ \pi\ r_1 r_2 r_3$. In Cartesian coordinates, the description of the change in size requires specification of the change in length of line segments in the three coordinate directions.

The change in absolute length is an inadequate measure of the deformational state of a line segment, because for a given change in length, the intensity of the change is much greater for a short line segment than for a long one. Thus the lengthening is expressed as a proportion of the original line length. Two measures in common use are the stretch s_n and the extension e_n, which was introduced at the beginning of Chapter 9. The subscript n indicates that the stretch or extension is measured in a direction parallel to a specified unit vector **n**. The stretch s_n is the ratio of the deformed length ℓ of a material line segment to its undeformed length L.

$$s_n \equiv \frac{\ell}{L} \qquad (15.1)$$

(We often use upper-case letters when referring to the undeformed state and lower-case letters when referring to the deformed state.) The extension e_n of a material line segment is the ratio of its change in length, ΔL, to its initial length L, where the change in length is the final length minus the initial length.[1]

$$e_n \equiv \frac{\ell - L}{L} = \frac{\Delta L}{L} \qquad (15.2)$$

[1] Note that with the sign conventions we have adopted, a positive value for extension measures a lengthening, whereas a positive value for stress measures a compression. We thus end up with a positive stress causing a negative extension. This incompatibility does not arise with the engineering sign convention for stress, which is why it is generally used in analytic applications of continuum mechanics.

Box 15.1 A More Quantitative View of Strain

A homogeneous transformation of any material point from the undeformed state to the deformed state is represented mathematically by a linear relationship between the coordinates of any point in the undeformed state (X_1, X_3) and its coordinates in the deformed state (x_1, x_3), where we use upper-case letters to describe the undeformed state and lower-case letters to describe the deformed state. If we restrict our analysis to plane deformation, the general form of such a transformation is

$$x_1 = AX_1 + BX_3 + C \qquad x_3 = DX_1 + EX_3 + F \quad (15.1.1)$$

where A, B, C, D, E, and F are constants. The parts of the transformation defined by C and F are the same for all particles, and therefore these constants describe a rigid-body translation. If any or all of these constants vary with time, then these equations describe the motion of the material particles.

The equations say that given the original location of any material particle in the undeformed state (X_1, X_3), we can calculate its final location in the deformed state (x_1, x_3). The equations may be solved for X_1 and X_3 so that given the deformed location of a material particle (x_1, x_3), we can also calculate its original location (X_1, X_3). These equations define the inverse transformation.

$$X_1 = ax_1 + bx_3 + c \qquad X_3 = dx_1 + ex_3 + f \quad (15.1.2)$$

where

$$a \equiv \frac{E}{AE - BD} \qquad b \equiv \frac{-B}{AE - BD} \qquad c \equiv \frac{BF - CE}{AE - BD}$$
$$d \equiv \frac{-D}{AE - BD} \qquad e \equiv \frac{A}{AE - BD} \qquad f \equiv \frac{DC - AF}{AE - BD} \quad (15.1.3)$$

and where, again, c and f describe a rigid body translation.

As examples of such a transformation and its inverse, the following equations describe a pure shear, which transforms a square with sides parallel to the principal coordinates into a rectangle (Figure 15.9B)

$$\begin{aligned} x_1 &= AX_1 & x_3 &= (1/A)X_3 \\ X_1 &= (1/A)x_1 & X_3 &= Ax_3 \end{aligned} \quad (15.1.4)$$

A simple shear, which transforms a square into a parallelogram (Figure 15.11B), and its inverse are described by

$$\begin{aligned} x_1 &= X_1 + BX_3 & x_3 &= X_3 \\ X_1 &= x_1 - Bx_3 & X_3 &= x_3 \end{aligned} \quad (15.1.5)$$

When the constants A in Equation (15.1.4) and B in Equation (15.1.5) are linear functions of time, the motions are steady and these equations describe progressive pure shear and progressive simple shear, respectively (see Section 15.4).

With Equations (15.1.2), it is easy to show that a homogeneous deformation transforms a circle into an ellipse. A circle of unit radius in the undeformed state is represented by the equation

$$(X_1)^2 + (X_3)^2 = 1 \quad (15.1.6)$$

If we substitute for X_1 and X_3 from Equations (15.1.2), we find the locus in the deformed state of all material particles that lie on the circle in the undeformed state. Because a rigid-body translation does not contribute to the strain, we assume $c = f = 0$. Then, making the substitution, we find

$$(a^2 + d^2)(x_1)^2 + 2(ab + de)x_1x_3 + (b^2 + e^2)(x_3)^2 = 1 \quad (15.1.7)$$

Equation (15.1.7) is the equation of an ellipse with its principal axes tilted with respect to the coordinate axes, and it is, in fact, the *strain ellipse*.

The components of the strain tensor are related to the displacement vectors for the material particles. A displacement vector connects the position of a particle in the undeformed state to its position in the deformed state. The vector and its components (U_1, U_3) parallel to the X_1 and X_3 coordinate axes are (Figure 15.1.1A)

$$\mathbf{U} \equiv \mathbf{x} - \mathbf{X} \quad (15.1.8)$$

$$U_1 = x_1 - X_1 \qquad U_3 = x_3 - X_3 \quad (15.1.9)$$

When a material deforms, the displacement vectors for two neighboring material points are different. If they were the same, the "deformation" would be a rigid body motion. The difference in these displacement vectors therefore describes the deformation. Thus we consider two neighboring points A and B that are displaced by the deformation to a and b, respectively. The displacement vectors for the two points are $\mathbf{U}^{(A)}$ and $\mathbf{U}^{(B)}$, and the difference between them is $d\mathbf{U}$ (Figure 15.1.1B). The material line segment $d\mathbf{X}$ connecting A to B is deformed into $d\mathbf{x}$ connecting a to b. The change in that line segment due to the deformation $\Delta d\mathbf{X}$ is also described by the vector $d\mathbf{U}$ (Figure 15.1.1B). Thus,

$$d\mathbf{U} \equiv \mathbf{U}^{(B)} - \mathbf{U}^{(A)} = \Delta d\mathbf{X} \equiv d\mathbf{x} - d\mathbf{X} \quad (15.1.10)$$

The relationship between the first and last terms in this equation is just the differential of Equation (15.1.8).

We can consider the components dX_1 and dX_3 of the line segment $d\mathbf{X}$ to be two material line segments that are initially perpendicular to each other and parallel to the coordinate axes X_1 and X_3 respectively. If we restrict our analysis to infinitesimal strain,

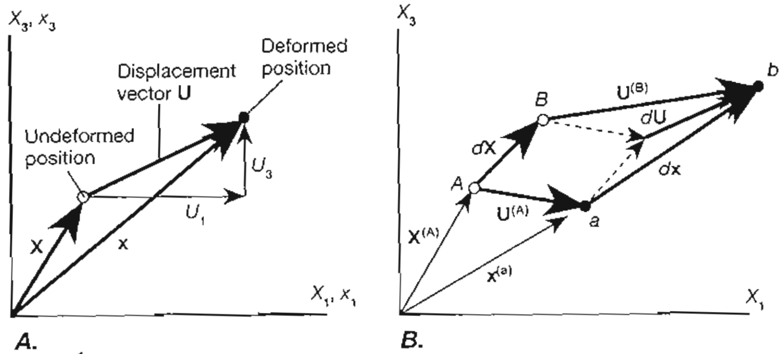


Figure 15.1.1 The displacement vector. *A.* The displacement vector connects the position of a material particle in the undeformed state to its position in the deformed state. *B.* If a material is deformed, the displacement vectors for two neighboring points are different. Point *A* is deformed to the position *a*; *B* is deformed to the position *b*. The difference in the displacement vectors dU describes the deformation of the material.

characterized by the conditions $dU_1 \ll 1$ and $dU_3 \ll 1$, the displacement associated with each of these line segments due to the deformation can be expressed using Equation (15.1.10) and the chain rule of differentiation for $d\mathbf{U}$

$$\Delta d\mathbf{X} = \Delta dX_1 + \Delta dX_3 = d\mathbf{U} = \frac{\partial \mathbf{U}}{\partial X_1} dX_1 + \frac{\partial \mathbf{U}}{\partial X_3} dX_3 \tag{15.1.11}$$

Thus the changes ΔdX_1 and ΔdX_3 in each of the line segments due to the deformation is given in terms of the components of the displacement vector $\mathbf{U}$ by (Figure 15.1.2).

$$\begin{aligned} \Delta dX_1 &= \frac{\partial \mathbf{U}}{\partial X_1} dX_1 = \frac{\partial U_1}{\partial X_1} dX_1 + \frac{\partial U_3}{\partial X_1} dX_1 \\ \Delta dX_3 &= \frac{\partial \mathbf{U}}{\partial X_3} dX_3 = \frac{\partial U_1}{\partial X_3} dX_3 + \frac{\partial U_3}{\partial X_3} dX_3 \end{aligned} \tag{15.1.12}$$

For each of the material line segments dX_1 and dX_3, the extensional strains are labeled e_{11} and e_{33} respectively, and each one is the change in length divided by the initial length, as defined in Equation (15.2). For dX_1, for example, the change in length is $(\partial U_1/\partial X_1)\, dX_1$, and the initial length is dX_1 (Figure 15.1.2). Similar relations hold for dX_3. Thus

$$\begin{aligned} e_{11} &\equiv \frac{1}{dX_1}\left[\frac{\partial U_1}{\partial X_1} dX_1\right] = \frac{\partial U_1}{\partial X_1} \\ e_{33} &\equiv \frac{1}{dX_3}\left[\frac{\partial U_3}{\partial X_3} dX_3\right] = \frac{\partial U_3}{\partial X_3} \end{aligned} \tag{15.1.13}$$

The shear strain of dX_1 relative to dX_3 and vice versa are labeled e_{13} and e_{31}, respectively, and are defined in Equation (15.7) to be half the tangent of the shear angle $\psi_{13} = \psi_{31} = \psi = \alpha + \beta$. For very small strains, $\alpha \ll 1$ and $\beta \ll 1$, and the standard trigonometric identity for the tangent of the sum of two angles gives

$$\tan \psi \equiv \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \approx \tan \alpha + \tan \beta \tag{15.1.14}$$

because the product $\tan \alpha \tan \beta$ is negligibly small. The tangent of an angle is the length of the side opposite the angle divided by the length of the adjacent side. For infinitesimal strains, the side opposite the angle α is approximately $(\partial U_1/\partial X_3)dX_3$, and the adjacent side is dX_3 (Figure 15.1.2). Similar relationships hold for the angle β. Thus we have

$$\begin{aligned} \tan \alpha &\approx \frac{1}{dX_3}\left[\frac{\partial U_1}{\partial X_3} dX_3\right] = \frac{\partial U_1}{\partial X_3} \\ \tan \beta &\approx \frac{1}{dX_1}\left[\frac{\partial U_3}{\partial X_1} dX_1\right] = \frac{\partial U_3}{\partial X_1} \end{aligned} \tag{15.1.15}$$

(Continued)

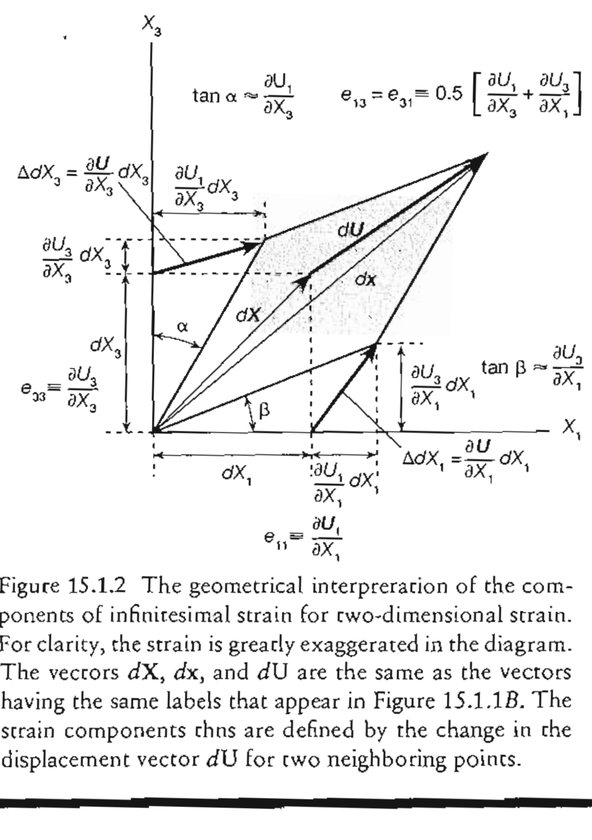


Figure 15.1.2 The geometrical interpretation of the components of infinitesimal strain for two-dimensional strain. For clarity, the strain is greatly exaggerated in the diagram. The vectors $d\mathbf{X}$, $d\mathbf{x}$, and $d\mathbf{U}$ are the same as the vectors having the same labels that appear in Figure 15.1.1*B*. The strain components thus are defined by the change in the displacement vector $d\mathbf{U}$ for two neighboring points.

Box 15.1 *(Continued)*

Then using the definition of the shear strain (Equation 15.7) with Equations (15.1.14) and (15.1.15) gives

$$e_{13} = e_{31} = 0.5 \tan \psi \approx 0.5\left(\frac{\partial U_1}{\partial X_3} + \frac{\partial U_3}{\partial X_1}\right) = 0.5\left(\frac{\partial U_3}{\partial X_1} + \frac{\partial U_1}{\partial X_3}\right) \qquad (15.1.16)$$

These relations for the extensions and shear strains associated with the material line segments dX_1 and dX_3 are the components of the infinitesimal strain tensor. In shorthand component notation, we summarize Equations (15.1.13) and (15.1.16) by

$$e_{k\ell} \equiv 0.5\left(\frac{\partial U_k}{\partial X_\ell} + \frac{\partial U_\ell}{\partial X_k}\right), \qquad k, \ell = 1, 2, 3 \quad (15.1.17)$$

This expression for $e_{k\ell}$ remains exactly the same if k and ℓ are interchanged, which shows that $e_{k\ell} = e_{\ell k}$ and that the strain tensor is a symmetric tensor (compare Equation 15.12). Thus $e_{k\ell}$ is the symmetric part of the displacement gradient tensor $\partial U_k/\partial X_\ell$.

The antisymmetric part of the displacement gradient tensor can be shown to be the infinitesimal rotation tensor, defined by

$$r_{k\ell} \equiv 0.5\left(\frac{\partial U_k}{\partial X_\ell} - \frac{\partial U_\ell}{\partial X_k}\right), \qquad k, \ell = 1, 2, 3 \quad (15.1.18)$$

The antisymmetric character of $r_{k\ell}$ is evident from this equation, because interchanging the subscripts k and ℓ gives the relation

$$r_{k\ell} = -r_{\ell k} \qquad (15.1.19)$$

Components on the principal diagonal of the matrix $r_{k\ell}$ must therefore be zero. In two-dimensional strain, there is only one independent off-diagonal component $r_{13} = -r_{31}$. Thus from Equations (15.1.15) and (15.1.18) we can see that

$$r_{13} \approx 0.5(\tan \alpha - \tan \beta) \qquad (15.1.20)$$

For very small angles, the tangent of the angle is approximately equal to the angle measured in radians, so we can write

$$r_{13} \approx 0.5(\alpha - \beta) \qquad (15.1.21)$$

Thus r_{13} is half the difference in the components of the shear angle, and $r_{k\ell}$ is thus a measure of the net rotation of the material line segment $d\mathbf{X}$.

The displacement components (U_1, U_3) can be expressed solely in terms of the coordinates of the material point in the undeformed state by substituting Equations (15.1.1) into (15.1.9), assuming the rigid translations are zero ($C = F = 0$)

$$U_1 = (A-1)X_1 + BX_3 \qquad U_3 = DX_1 + (E-1)X_3 \qquad (15.1.22)$$

Using Equations (15.1.22) in (15.1.17), we find the values of the strain components in terms of the constants that define the motion of the material particles:

$$\begin{bmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{bmatrix} = \begin{bmatrix} (A-1) & 0.5(B+D) \\ 0.5(D+B) & (E-1) \end{bmatrix} \qquad (15.1.23)$$

As indicated above, the relationships given here are correct only for very small strains. The analysis of large strains is considerably more complex, although this geometric interpretation of the strain components remains intuitively useful.

For a line segment of arbitrary orientation in the undeformed state, given by the angle θ with respect to the principal coordinate axis $\hat{X}_1$, it can be shown that the extension and the shear strain for infinitesimal plane strain are given in terms of the principal extensions by

$$e_n = \hat{e}_1 \cos^2\theta + \hat{e}_3 \sin^2\theta \qquad e_s = (\hat{e}_1 - \hat{e}_3)\sin\theta\cos\theta \qquad (15.1.15)$$

These equations are identical in form to Equations (8.36), which we found for the stress components, and the mathematical characteristics of the stress and the infinitesimal strain tensors are identical, including the possibility of deriving a Mohr circle for infinitesimal strain.

The relationships for large deformations are somewhat more complex, but a Mohr circle that is useful in solving strain problems can nevertheless be defined for large strains. We refer the reader to books containing more quantitative analyses (see the works by Means, Ramsay and Huber, and Eringen in the list of additional readings at the end of this chapter).

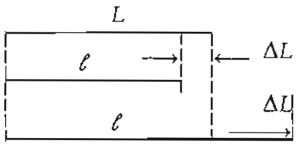
Comparing Equations (15.1) and (15.2) shows that these two measures of extensional strain are related:

$$e_n = \frac{\ell}{L} - \frac{L}{L} = s_n - 1 \qquad (15.3)$$

Values of $s_n > 1$ and of $e_n > 0$ represent increases in the length of material lines, and values where $0 < s_n < 1$ and $e_n < 0$ represent decreases in length (Table 15.1).

Other measures are also used, including the quadratic elongation and the natural strain. The quadratic elongation is simply the square of the stretch, and it is often given the symbol λ, although some authors use this symbol to designate the stretch. The natural strain $\bar{\varepsilon}_n$, also called the logarithmic strain, is the integral of all the infinitesimal increments of extension required to make up the deformation, where the reference length

Table 15.1 Extensional Strain of a Material Line

		Length Change ΔL	Stretch $s_n \equiv \ell/L$	Extension $e_n \equiv (\ell - L)/L$
Undeformed		$\Delta L = 0$	$s_n = 1$	$e_n = 0$
Shortened		$\Delta L = \ell - L < 0$	$0 < s_n < 1$	$e_n < 0$
Lengthened		$\Delta L = \ell - L > 0$	$s_n > 1$	$e_n > 0$

for each increment in length $d\ell$ is taken to be the instantaneous deformed length ℓ.

$$\bar{\varepsilon}_n \equiv \int_L^{\ell_f} \frac{d\ell}{\ell} = \ln\left(\frac{\ell_f}{L}\right) = \ln s_n \qquad (15.4)$$

where L is the initial length, ℓ_f is the final length, and ln indicates the natural logarithm. Notice that the natural strain is the natural logarithm of the stretch. The natural strain is sometimes convenient for discussion of strain history (see Figure 15.20). It also provides a symmetric measure of shortening and lengthening.[2] The time derivative of the natural strain is also often used as a measure of the strain rate (see Box 18.1).

Volumetric Strain

We can now consider measures of the volumetric strain, which we refer to as the volumetric streteh (s_v) and the volumetric extension[3] (e_v). If the undeformed volume is V and the deformed volume is v,

$$s_v \equiv \frac{v}{V} \qquad e_v \equiv \frac{v - V}{V} = \frac{\Delta V}{V} = s_v - 1 \qquad (15.5)$$

A rectangular block thar undergoes only volumetric strain has undeformed sides (L_1, L_2, and L_3) and deformed sides (ℓ_1, ℓ_2, and ℓ_3). The volumetric stretch is

$$s_v = \frac{\ell_1 \ell_2 \ell_3}{L_1 L_2 L_3}$$

$$s_v = s_1 s_2 s_3 = (e_1 + 1)(e_2 + 1)(e_3 + 1) \qquad (15.6)$$

We consider further aspects of volumetric strain in the next section.

Shear Strain

A body can also change shape without changing volume. For example, a cube can deform into a rhombohedron, or a sphere into an ellipsoid. Changes in shape are described by the changes in the angle between pairs of lines that are intially perpendicular (Figure 15.2). The change in angle is called the shear angle ψ, and the shear strain e_s is defined by

$$e_s \equiv 0.5 \tan \psi \qquad (15.7)$$

As defined here, e_s is the tensor shear strain. It differs from another common measure of the shear strain, the engineering shear strain γ, by a factor of 2 ($\gamma \equiv \tan \psi = 2e_s$). For two material line segments originally oriented along the positive coordinate directions (Figure 15.2*A*), a decrease in angle between the two lines is considered a positive shear strain (Figure 15.2*B*, *C*) and an increase in angle is a negative shear strain (Figure 15.2*D*, *E*). Both γ and e_s increase from 0 in the unstrained state to ∞, where $\psi = 90°$ (Figure 15.2*F*).

15.2 The State of Strain: The Strain Ellipsoid and the Strain Tensor

The Strain Ellipsoid

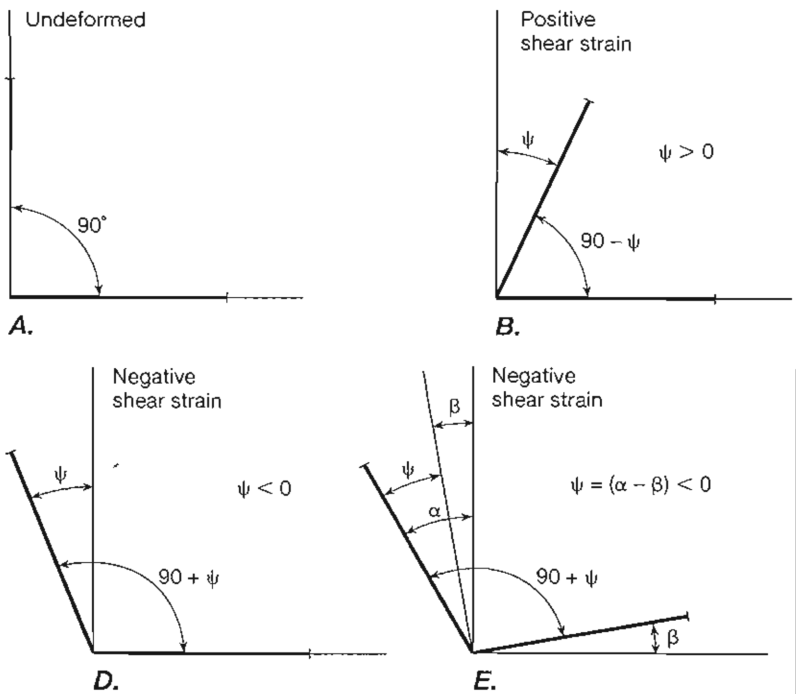
We know the state of strain at a point if, for a material line of any orientation, we can determine its extension, as well as its shear strain with respect to any other line initially perpendicular to it. Any homogeneous strain always deforms a material sphere into an ellipsoid called the strain ellipsoid (Figures 14.1 and 14.2*A*) or, in plane strain, a material circle into the strain ellipse (see Box 15-1).

The stretch, extension, and shear strain all have a simple geometric interpretation related to the strain ellipsoid. We describe these relationships here for two dimensions, but they are essentially the same when extended to three dimensions.

Assume that a material circle in the undeformed stare has a radius $R = 1$ (Figure 15.3*A*). After the deformation, any radius of the circle is transformed into a radius r of the strain ellipse whose length varies with orientation. Although R and r are lines made up of the same material points, they differ in length and orientation because of the deformation. If we snperimpose the original unit circle on the strain ellipse (Figure 15.3*A*), we can see how much any radius of the strain

[2] For example, for a line segment stretched to twice its initial length and one shortened to half its initial length, $s_n = 2$ and 0.5, and $e_n = 1$ and 0.5, but $\bar{\varepsilon}_n = 0.693$ and -0.693, respectively.

[3] The volumetric extension is commonly given the symbol Δ and called the dilation, or even the dilatation. We reserve Δ to indicate the change in a variable.



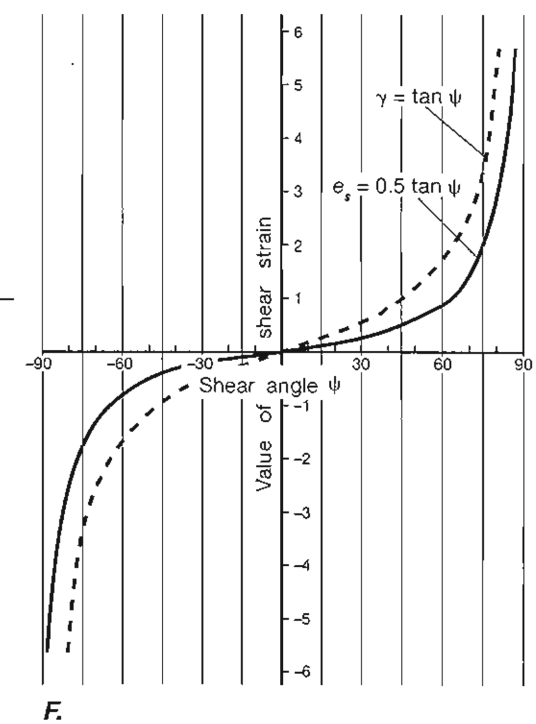


Figure 15.2 The tensor shear strain $e_s = 0.5 \tan \psi$ and the engineering shear strain $\gamma = \tan \psi$ of a material line, where ψ is the shear angle. *A.* The undeformed state. Shear of a material line is defined with reference to another material line initially normal to the first. *B* and *C.* Definition of a positive shear strain: $(90 - \psi) < 90$. *D* and *E.* Definition of a negative shear strain: $(90 - \psi) > 90$. *F.* Tensor and engineering shear strains as a function of shear angle ψ.

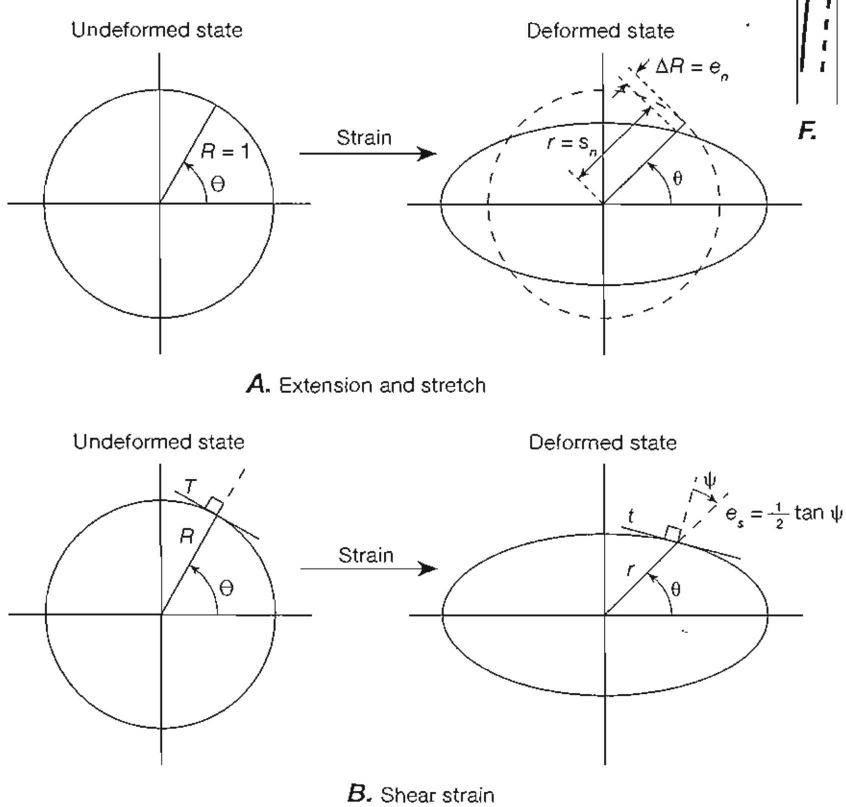


Figure 15.3 The relationship of the stretch, extension, and shear strain to the geometry of the strain ellipse. *A.* A homogeneous strain transforms the unit circle into an ellipse. An undeformed radius $R = 1$ is transformed into a deformed radius r, which has a different length and orientation. The stretch is the length of the radius of the ellipse, and the extension is the difference in radius between the initial unit circle and the ellipse. *B.* The shear strain is determined from the change in angle between a radius and a tangent at the end of the radius. The two lines are perpendicular on the circle but not, in general, on the ellipse. The change in angle ψ defines the shear strain for that pair of lines.

ellipse has been shortened or lengthened. Using the definitions of the stretch (Equation 15.1) and the extension (Equation 15.2) and the fact that $R = 1$, we find that

$$s_n = \frac{r}{R} = r \qquad e_n = \frac{r - R}{R} = \frac{\Delta R}{R} = \Delta R \qquad (15.8)$$

Thus for the deformation of the unit circle, the radius of the strain ellipse is the stretch, and the difference between the radius of the ellipse and that of the unit circle is the extension.

The shear strain of a line is determined with reference to another line initially normal to it. On a circle, the line T drawn perpendicular to any radius R at its end point is tangent to the circle (Figure 15.3*B*). After deformation, the lines T and R are transformed into the lines t and r, respectively. Although r and t are no longer perpendicular in the deformed state, t is still tangent to the ellipse at the end point of the radius. Accordingly, any radius and the associated tangent to the strain ellipse define the angle between two material lines that were perpendicular in the undeformed state. The change in angle ψ is thus easily constructed (Figure 15.3*B*), and it is a measure of the shear strain for that pair of lines.

The Strain Tensor

The strain ellipsoid is a complete representation of the state of strain at a point. We can describe that state if we know the extension and the two shear strains for each of only three material line segments that were mutually orthogonal in the undeformed state. We consider the volumetric and the shear components of the strain separately.

For an orthogonal coordinate system (X_1, X_2, X_3) in the undeformed state, the extension of a material line segment of length L_1 initially parallel to X_1, for example, is (Figure 15.4*A*).

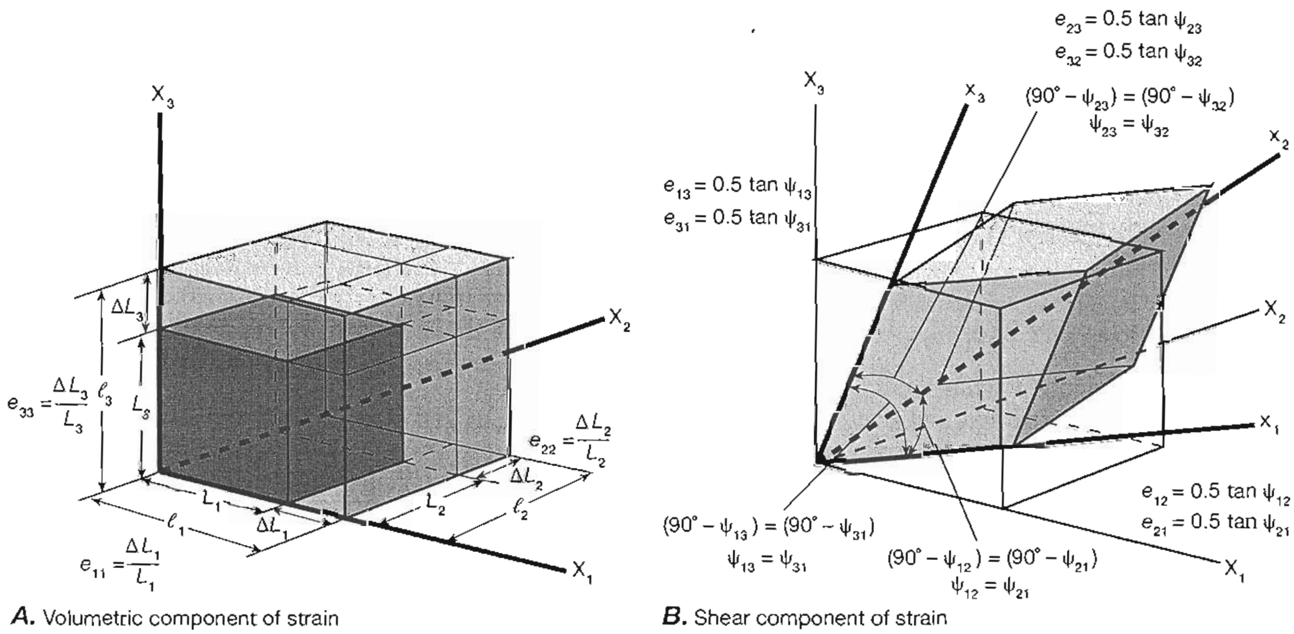
$$e_{11} = \frac{\Delta L_1}{L_1} \qquad (15.9)$$

where the first subscript on e_{11} indicates that the line is initially parallel to X_1, and the second subscript indicates that the change in length is also parallel to X_1. Similar relations define the extensions e_{22} and e_{33} for material lines initially parallel to X_2 and X_3 respectively (Figure 15.4*A*).

For the shear component of the strain, material lines initially parallel to X_1, X_2, and X_3 are, after deformation, parallel to x_1, x_2, and x_3 respectively (Figure 15.4*B*). The two shear strain components for the material line parallel to x_1 are e_{12} and e_{13},

$$e_{12} = 0.5 \tan \psi_{12} \qquad e_{13} = 0.5 \tan \psi_{13} \qquad (15.10)$$

In each case the first subscript indicates that the shear strain is for the line initially parallel to X_1, and the second subscript indicates that the shear strain is determined relative to a line initially parallel to X_2 and to X_3, respectively (Figure 15.4*B*). Each angle ψ_{12} and ψ_{13} is the difference between 90° and the deformed angle



A. Volumetric component of strain ***B.*** Shear component of strain

Figure 15.4 Geometric significance of the strain tensor components in three dimensions. *A*. Volumetric part of the strain. The small cube increases in volume to the larger cube by the equal lengthening of all sides of the cube. *B*. Shear part of the strain. The shear strain describes the change in shape from a cube into a rhombohedron (shaded). x_1, x_2, and x_3 are parallel to the deformed edges of the rhombohedron. All the tensor shear strain components are defined by three independent angles $\psi_{12} = \psi_{21}$, $\psi_{13} = \psi_{31}$, $\psi_{23} = \psi_{32}$.

$x_1 \wedge x_2$ and $x_1 \wedge x_3$, respectively. The comparable strain components for the material line segment initially parallel to X_2 are e_{21} and e_{23}; for the material line segment initially parallel to X_3, they are e_{31} and e_{32}.

Thus there are a total of nine strain components. The strain components for each material line are written in a separate row, forming an ordered array.

$$e_{k\ell} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \qquad (15.11)$$

PRINCIPAL DIAGONAL

The components on the principal diagonal of the array, which have both subscripts the same, are the extensions (Figure 15.4*A*). The off-diagonal components, which have two different subscripts, are the shear strains (Figure 15.4*B*). This array of strain components represents the strain tensor, which provides enough information for us to calculate the extension and shear strain for a line segment of any specified orientation (see Box 15.1).[4]

The strain tensor is symmetric about the principal diagonal, because for a given pair of material lines initially parallel to X_1 and X_2, for example, the shear angle (ψ_{12}) of X_1 with respect to X_2 is the same as the shear angle (ψ_{21}) of X_2 with respect to X_1 (Figure 15.4*B*). Thus

$$e_{12} = e_{21} \qquad e_{23} = e_{32} \qquad e_{31} = e_{13} \qquad (15.12)$$

and there are only six independent strain components in three-dimensional strain. Thus the strain, like the stress, is a second-rank symmetric tensor.

For plane strain, we have $e_{21} = e_{22} = e_{23} = 0$, and by Equations (15.12), $e_{12} = e_{32} = 0$. Thus if we drop from Equation (15.11) all terms that necessarily become zero for plane strain, the plane strain tensor is represented by only four strain components, three of which are independent.

$$e_{k\ell} = \begin{bmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{bmatrix} \qquad (15.13)$$

Therefore, in order to describe the state of plane strain, we need only the extension and one shear strain for each of the two material lines that originally are parallel to X_1 and X_3, respectively.

[4] Our definitions of the tensor strain components are correct only for small strains. For large strains, additional nonlinear terms must be added to our definitions, which makes the theory more complex. Nevertheless, the results discussed hereafter are true for both small and large strains.

Principal Strains and Stretches

Parallel to the principal axes of the strain ellipsoid, the extensions and stretches are a maximum, minimax,[5] and minimum, which we designate[6]

$$\hat{e}_1 \geq \hat{e}_2 \geq \hat{e}_3 \quad \text{and} \quad \hat{s}_1 \geq \hat{s}_2 \geq \hat{s}_3 \qquad (15.14)$$

Tangents to the ellipsoid at the ends of the principal radii are perpendicular to the radii (Figure 15.5), and these are the only points on the ellipsoid where this is true. Because these radii and tangents must have been perpendicular before deformation, the shear strains for those radii and tangents all must be zero. Thus if we define a set of principal coordinates parallel to the principal axes of the strain ellipsoid, the representation of the strain tensor reduces to a particularly simple form in which the extensions are the principal values, and the shear strains are zero. For three- and two-dimensional strains, respectively

$$e_{k\ell} = \begin{bmatrix} \hat{e}_1 & 0 & 0 \\ 0 & \hat{e}_2 & 0 \\ 0 & 0 & \hat{e}_3 \end{bmatrix} \qquad e_{k\ell} = \begin{bmatrix} \hat{e}_1 & 0 \\ 0 & \hat{e}_3 \end{bmatrix} \qquad (15.15)$$

It is very important to remember that in general the principal axes of finite strain are *not* parallel to the principal axes of stress. We discuss this further in Section 15.4 and in Chapter 18.

We now see that, for any general deformation, the volumetric stretch s_v (Equation 15.6) can be expressed in terms of the principal stretches and extensions as follows:

$$s_v = \hat{s}_1\hat{s}_2\hat{s}_3 = (\hat{e}_1 + 1)(\hat{e}_2 + 1)(\hat{e}_3 + 1) \qquad (15.16)$$

[5] $\hat{e}_2$ and $\hat{s}_2$ are each a minimax because each is a minimum in the $\hat{e}_1 - \hat{e}_2$ (or $\hat{s}_1 - \hat{s}_2$) plane and a maximum in the $\hat{e}_2 - \hat{e}_3$ (or $\hat{s}_2 - \hat{s}_3$) plane, which is perpendicular to the first.

[6] Consistent with our notation for stress, we use the circumflexes and a single subscript to indicate principal values. The subscript indicates the principal axis to which the extension or stretch is parallel.

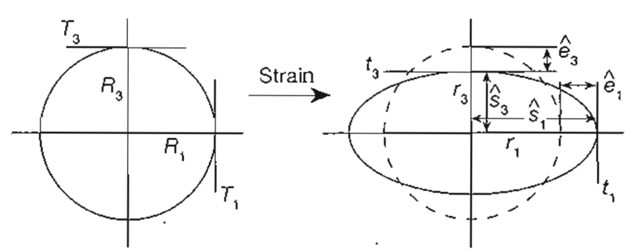


Figure 15.5 Representation of the principal stretches and the principal extensions on the strain ellipse formed from the unit circle. The shear strains are zero for the material lines parallel to the principal axes of strain, because the tangents at the ends of the principal radii are perpendicular to those radii both before and after deformation.

Although derived for the example of a deformed cube, Equations (15.16) are completely general.[7] In plane strain, $\hat{s}_2 = 1$ and $\hat{e}_2 = 0$, so Equation (15.6) reduces to

$$s_v = \hat{s}_1\hat{s}_3 = (\hat{e}_1 + 1)(\hat{e}_3 + 1) \qquad (15.17)$$

Thus the condition for constant-volume deformation is given for three-dimensional and plane strains, respectively, by

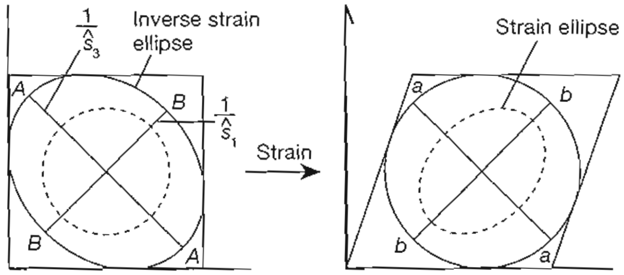
$$s_v = \hat{s}_1\hat{s}_2\hat{s}_3 = 1 \quad \text{and} \quad s_v = \hat{s}_1\hat{s}_3 = 1 \qquad (15.18)$$

The last equation implies

$$\hat{s}_1 = \frac{1}{\hat{s}_3} \qquad (15.19)$$

The Inverse Strain Ellipse

In analyzing large strains such as are common in ductilely deformed rocks, it may be more convenient to measure the stretches and shear strains of three material lines that are mutually perpendicular in the *strained* state, rather than in the unstrained state as we described above. This analysis requires a different strain ellipse called the inverse strain ellipse, which is the ellipse in the undeformed state that is transformed into a circle in the deformed state (Figure 15.6). The lengths of its principal axes are the inverse of the principal axes of the strain ellipse, and the material lines parallel to the principal axes of inverse strain in the undeformed state become parallel to the principal axes of strain in the deformed state. For the purposes of our descriptive discussion, however, we deal mostly with the strain ellipse.



A. Undeformed state **B.** Deformed state

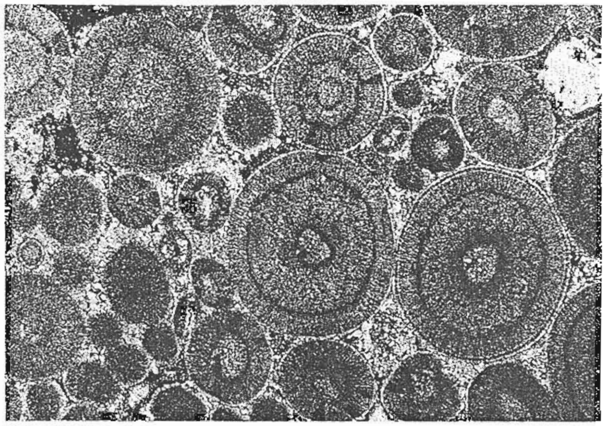
Figure 15.6 Definition of the inverse strain ellipse and its relationship to the strain ellipse. Solid lines show how the inverse strain ellipse in the undeformed state (part *A*) is transformed into a circle in the deformed state (part *B*). The dashed lines show how a circle in the undeformed state (part *A*) is transformed into the strain ellipse in the deformed state (part *B*). Material lines *A* and *B*, which are parallel to the principal axes of inverse strain ellipse in part *A*, are transformed by the deformation to lines *a* and *b*, which are parallel to the principal axes of the strain ellipse in part *B*. In general, *A* and *B* are not parallel to *a* and *b*, respectively.

Why Study Strain?

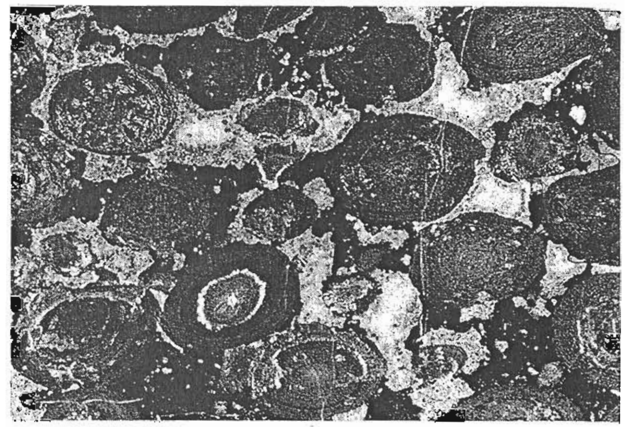
All this discussion of circles and ellipses may seem academic and far removed from the study of real rocks. It is not, however, because structures that are initially approximately circular or spherical are relatively common in some rock types. Where these rocks have been deformed, those structures provide a fascinating record of the distribution of strain throughout the rock. Ooids, for example, are small, almost spherical, pelletlike bodies common in limestones (Figure 15.7*A*), and they deform with the rock to record the shape and orientation of the strain ellipsoid (Figure 15.7*B*). Radiolaria and foraminifera, which are tiny spherical or disk-shaped fossils found in cherts or limestones, and alteration spots in slates (Figure 13.19*B*) may also serve as strain indicators. Other fossils, such as cephalopods and brachiopods, as well as pebbles and cobbles in conglomerates, (Figure 13.19*A*) can provide information about the strain, even though they are not originally spherical and may have an original preferred dimensional orientation in the undeformed rock (see Figure 14.2C). We discuss the significance of strain for interpreting the origin of structures in Chapter 16, and the measurement and observation of strain in deformed rocks in Chapter 17.

Some structures, such as folds and boudins, also record components of the strain. Consider, for example, a competent layer imbedded in an incompetent matrix. A variety of structures can develop (Figure 15.8). A set of folds develops if the layer is parallel to a principal axis of shortening and normal to an axis of lengthening (Figure 15.8*A*–*D*). Boudins develop if the layer is parallel to a principal axis of lengthening (Figure 15.8*C*–*F*). Two interfering sets of folds form if the layer is parallel to two principal directions of shortening and normal to an axis of lengthening (Figure 15.8*A*). Folds develop that are boudinaged parallel to the fold axis if the layer is perpendicular to a principal axis of lengthening, and the two principal axes parallel to the layer are axes of lengthening and shortening respectively (Figure 15.8*C*, *D*). Finally, tablet boudinage develops if the layer is parallel to two principal axes of lengthening and perpendicular to one of shortening (Figure 15.8*F*). Thus the orientation of the layer relative to the principal stretches is a major factor in determining what structures can develop.

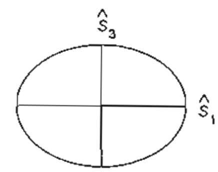
[7] We derive Equation (9.6) from the equation for e_v in Equations (15.5) by substituting for s_v from the second Equation (15.16), multiplying out the indicated product, and ignoring second- and third-order terms. The result is the sum of the components on the principal diagonal of the strain tensor matrix (Equation 15.11), which is a scalar invariant of the strain tensor (see the definition of the scalar invariants of the stress tensor in Equations 8.4.4 and 8.26) and hence is the same for the representation of strain in any coordinate system. That is, $e_v = \hat{e}_1 + \hat{e}_2 + \hat{e}_3 = e_{11} + e_{22} + e_{33}$.



A.



B.



Fignre 15.7 Ooids serve as strain markers in deformed limestone. *A.* An undeformed ooliric limesrone. *B.* A deformed oolitic limestone. The ratio of rhe principal srretches is $(\hat{s}_1/\hat{s}_3) \approx 1.5$. The larger ooids are approximately 4 mm in diameter.

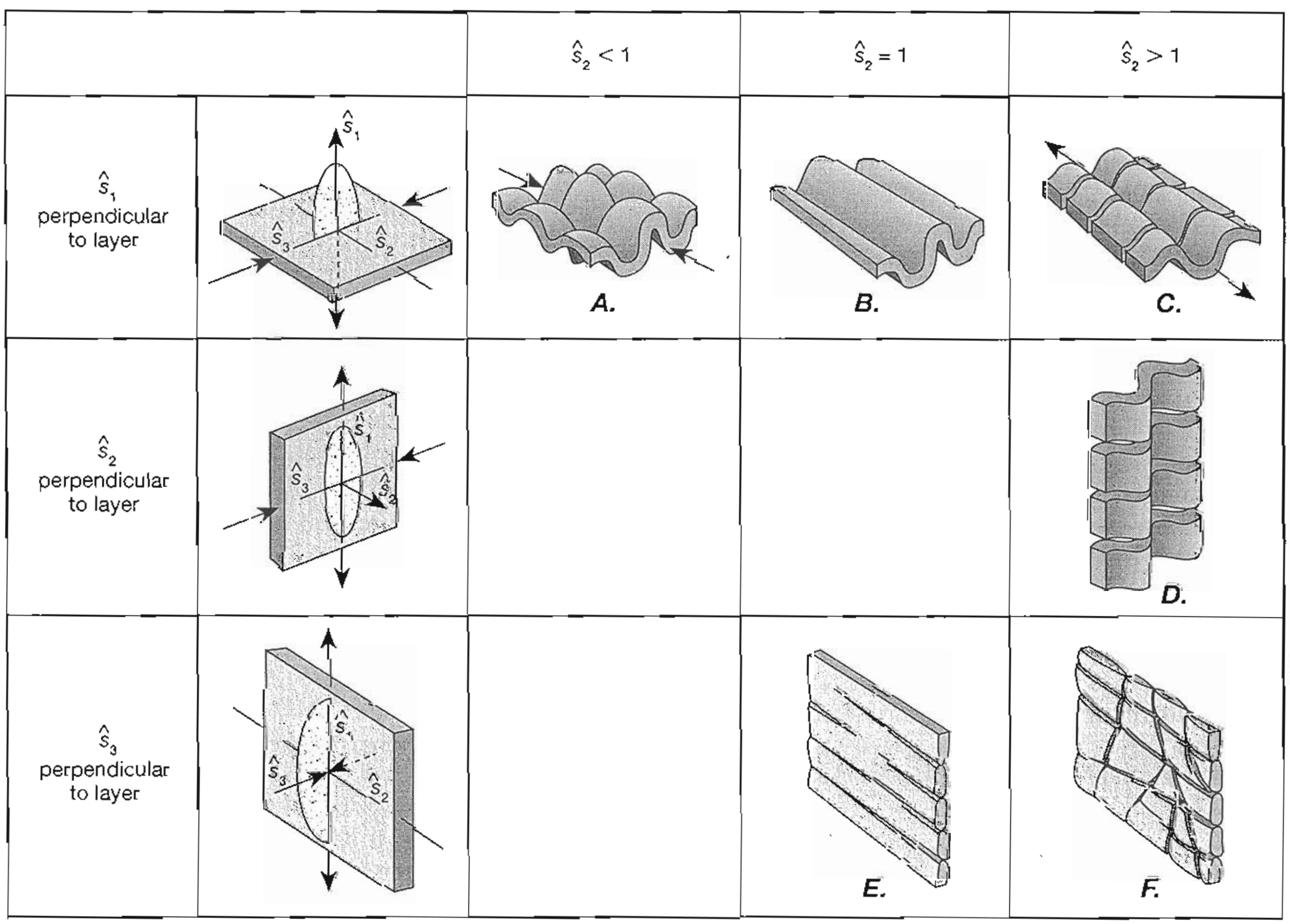


Figure 15.8 Structures that could develop in a competent layer imbedded in an incompetent matrix depend on the orientation of the layer relative to the principal stretches, and on the value of $\hat{s}_2$. In this diagram, we assume that lengthening the layer causes boudinage, shortening causes folding, and deformation is at constant volume, so thar $\hat{s}_1 > 1$, $\hat{s}_3 < 1$, and $\hat{s}_2$ can take on any value.

15.3 Examples of Homogeneous Strains

Various simple geometries of homogeneous strain are given specific names.

Pure strain is any strain for which the principal axes of strain are constant in orientation relative to the reference coordinate system. Thus the principal axes of strain and the principal axes of inverse strain are parallel. Strain geometries belonging to this class (and described below) include uniform dilation, pure shear, simple extension, simple flattening, and uniaxial strain.

Uniform dilation is a pure volumetric strain with no change in shape of the deforming body. A cube or a square is transformed into a body that is of the same shape but has either a larger dimension (uniform expansion) or a smaller dimension (uniform contraction). The same statement, of course, applies to both a sphere or a circle. The stretch has the same value in all directions, as does the extension, and the shear strains are zero in all directions; that is, $\psi = 0$ for all orientations of line. All material lines change length, but none changes orientation.

Pure shear is a constant-volume ($s_v = 1$) plane strain ($\hat{s}_2 = 1$) that changes the shape of the deforming body (Figure 15.9). Material lines parallel to the principal axes of strain do not rotate and experience no shear strain. Material lines of all other orientations in the plane of strain (the $\hat{s}_1$–$\hat{s}_3$ plane) are rotated toward $\hat{s}_1$. Two orientations of line in the plane of strain have the same length as their initial length; these are the lines of no finite extension. They divide the ellipse into sectors within which all radial lines are either shortened (sectors S in Figure 15.9C) or lengthened (sectors L), depending on their orientation.

Simple extension involves lengthening parallel to one principal axis of strain and axially symmetric shortening in all directions perpendicular to that axis. Simple flattening involves shortening parallel to one principal strain axis and axially symmetric lengthening in all directions perpendicular to that axis. The volume of the body in either case is not necessarily constant.

Uniaxial strain is characterized by having two of the principal stretches equal to 1. The third principal stretch may be either greater than 1 (uniaxial extension; Figure 15.10*A*) or less than 1 (uniaxial shortening, Figure 15.10*B*). Volume is not conserved. Lines perpendicular to the unique axis of stretch are unchanged in length. Lines in all other orientations are lengthened in uniaxial extension and shortened in uniaxial shortening.

Simple shear is a type of strain we discussed briefly at the beginning of Chapter 12 (Figure 12.1). It is a constant-volume ($s_v = 1$) plane strain ($\hat{s}_2 = 1$) whose characteristics resemble the shearing of a deck of cards; thus, for a homogeneous deformation, the side of the deck changes from a rectangle to a parallelogram (Figure 15.11*A*). It is not a pure strain, because the orientations of principal strain axes change with the magnitude of shear, and the principal axes of strain and of inverse strain are not parallel (Figure 15.11*B*). Displacement of all material particles is parallel to the shear plane (the x_1–x_2 plane in Figure 15.11) and all material lines are rotated except those parallel to the shear plane. There are two orientations of no finite extension in the plane of strain (the x_1–x_3 and $\hat{s}_1$–$\hat{s}_3$ plane), one of which is always parallel to the shear plane. These lines divide the strain ellipse into sectors of shortened radii (S in Figure 15.11C) and lengthened radii (L).

These states of strain are all special cases of the infinite variety of possible states. They have no special qualities that make them uniquely applicable to the interpretation of rock deformation, but they are used because the geometry of each is simple and well defined. An arbitrary deformation, however, can *always* be ex-

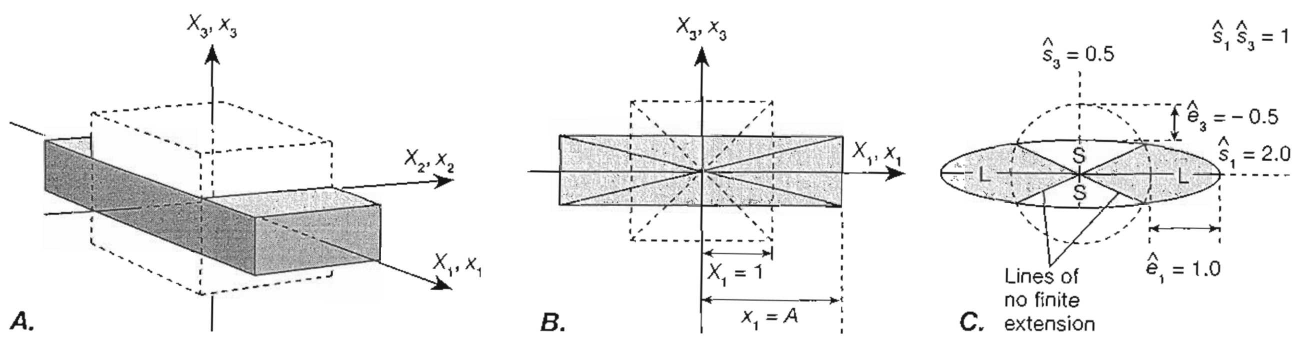


Figure 15.9 Pure shear: a constant-volume plane strain in which the principal axes of strain are not rotated by the deformation. *A*. Pure shear of a cube into a rectangular prism (shaded). *B*. Pure shear of a two-dimensional square to form a rectangle (shaded). The diagonals of the square are material lines that are rotated and stretched to become the diagonals of the rectangle; they are *not* the same as the lines of no finite elongation. *C*. Pure shear of a unit circle to form an ellipse. The lines of no finite extension divide the strain ellipse into sectors in which all radii are shortened (sectors S) and those in which all radii are lengthened (sectors L).

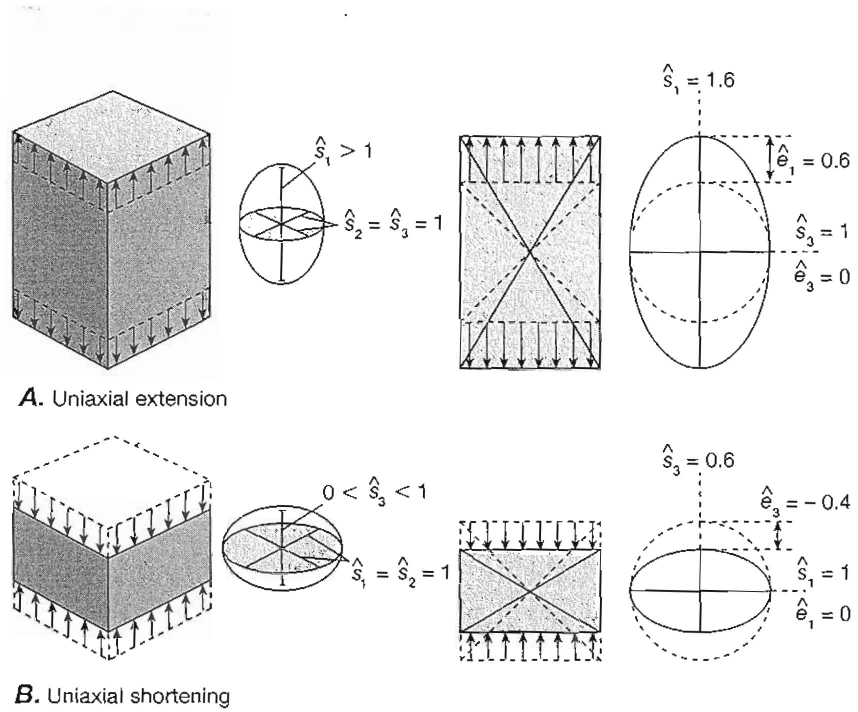


Figure 15.10 Uniaxial strain: two principal srretches are both equal to 1. Dashed lines indicate the undeformed state, solid lines the deformed state.

pressed as the sum of a pure strain that has stretches parallel to the axes of inverse strain (Figure 15.12*A*, *B*), a rigid rotation of the body that brings the principal axes of strain into the proper orientation (Figure 15.12*C*), and a rigid translation of the body that brings it into the proper location (Figure 15.12*D*). These components of the deformation can in principal be applied in any order. The net result of a simple shear strain (Figure 15.11), for example, can be reproduced by the sum of a pure shear (Figure 15.9) parallel to the axes of inverse strain, a rotation of the principal axes, and a translation (Figure 15.12). Other geometrically more complex deformations can be similarly reproduced.

15.4 Progressive Deformation

So far in our discussion, we have simply related the deformed state ro the undeformed state, without implying anything about the intermediate strain states that develop during the deformation. In rocks, we generally can observe only the final strained state and must infer the initial undeformed state. The history of the deformation is also of great inrerest, and in some cases it is recorded by features in deformed rocks. Understanding the consequences of different strain paths can provide insight that is useful in interpreting strain in rocks.

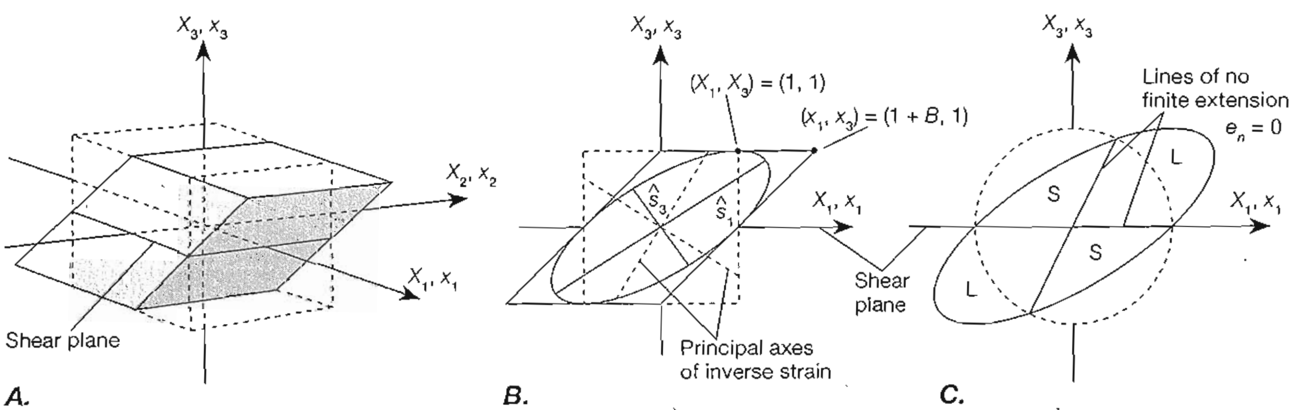


Figure 15.11 Simple shear: a constant-volume plane strain in which all material particles are displaced strictly parallel ro the shear plane. Dashed lines indicate the undeformed state, solid lines rhe deformed state. *A*. Simple shear of a cube. *B*. Simple shear in two dimensions of a square. The principal axes of inverse strain in the undeformed state are dashed; the principal axes of srrain in the deformed state are solid. Material lines parallel to the axes of inverse strain are rotated by the deformation into parallelism with the principal axes of srrain. C. Lines of no finite exrension in the strain ellipse divide the ellipse into sectors of shortened (S) and lengthened (L) radii of the ellipse.

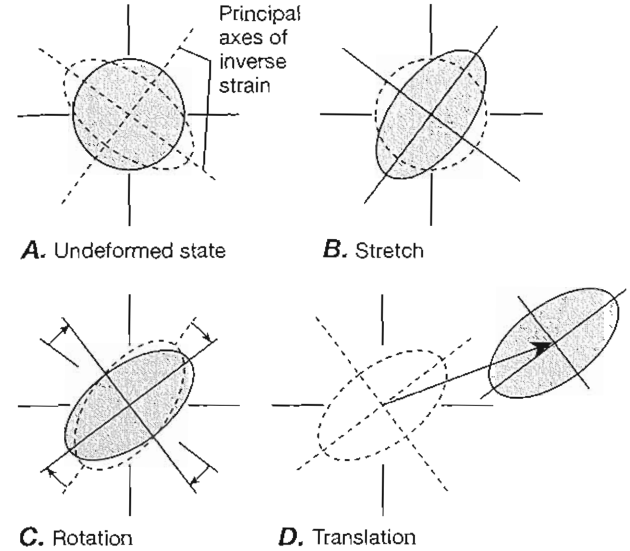


Figure 15.12 Decomposition of an arbitrary homogeneous srrain into a pnre strain, a rigid rotation, and a rigid translation. These componenrs may be applied in any seqnence. *A.* The nndeformed state, showing the unir circle, the inverse srrain ellipse (dashed), and rhe principal axes of inverse strain. *B.* Stretches are imposed parallel to the principal axes of inverse srrain to reproduce the final shape of rhe strain ellipse. The inverse strain ellipse becomes a circle. *C.* Rigid-body rotation brings the principal axes into the correct final orientation. *D.* Rigid-body rranslarion brings the body inro the correct final location.

We refer to the nonrigid motion of a body as a **progressive strain** or **progressive deformation**, and we can describe the motions of all material particles in the body by describing the deformed position of the particles as a function of their original position and of time (see Box 15.1).

Structures such as folds, boudins, foliations, and lineations develop in rock in response to progressive deformations. Folds and boudins develop in material layers in the rock, such as sedimentary layers, cross-cutting veins, or dikes. Most spaced foliations are also defined by material surfaces. Therefore, in order to understand the relationship between such structures and the principal axes of strain, we investigate what happens to material lines of various orientations during different progressive plane deformations.

We can conceptualize the geometry of the progressive deformation by stopping it, marking a material circle on the body, and allowing the deformation to continue for a unit increment of time. The ellipse formed from the circle represents the increment of strain for that increment of time and is therefore called the **incremental strain ellipse**. Thus the incremental extension ε_n, the incremental shear strain ε_s, and the incremental stretch ζ_n (the Greek letter zeta) are defined in terms of the instantaneous length of a material line ℓ, its incremental change $d\ell$, and the incremental shear angle $d\psi$ of two instantaneously perpendicular lines.

$$\varepsilon_n \equiv \frac{d\ell}{\ell} \qquad \varepsilon_s \equiv 0.5 \tan d\psi \qquad \zeta_n \equiv \frac{\ell + d\ell}{\ell} \tag{15.20}$$

The incremental strain ellipse is represented by the incremental strain tensor $\varepsilon_{k\ell}$, which has the same properties as the infinitesimal strain tensor.[8] The half-lengths of the principal axes are the principal incremental stretches, $\zeta_1 \geq \zeta_2 \geq \zeta_3$. If the incremental strain ellipse is constant for every unit increment in time, the motion of the material particles is called a **steady motion**.

To illustrate the effects of different motions on material lines, consider two special steady motions: progressive pure shear and progressive simple shear. The particle paths during these progressive deformations are shown in Figure 15.13*A* and *B*, respectively. (See Equations (15.1.4) and (15.1.5) in Box 15.1, for the quanti-

[8] Because the incremental strain ellipse represents the strain in a unit increment of time, it is similar to the strain rate tensor (see Box 18.1). Note that the natural strain $\bar{\varepsilon}_n$ is the integral of the incremental strain over time (see Eqnation 15.4).

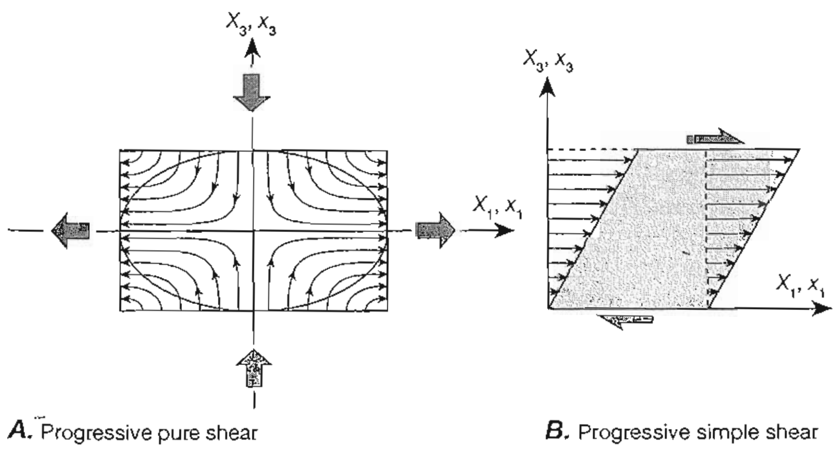


Figure 15.13 Parricle motions during two progressive deformations. *A.* Particle motions during progressive pure shear. The lines with the arrowheads are parallel ro rhe velociry vectors of rhe parricles in the body. *B.* Particle morions during progressive simple shear are all stricrly parallel to the shear plane (X_1 direction). The velociry varies linearly wirh disrance normal ro the shear plane (X_3 direcrion).

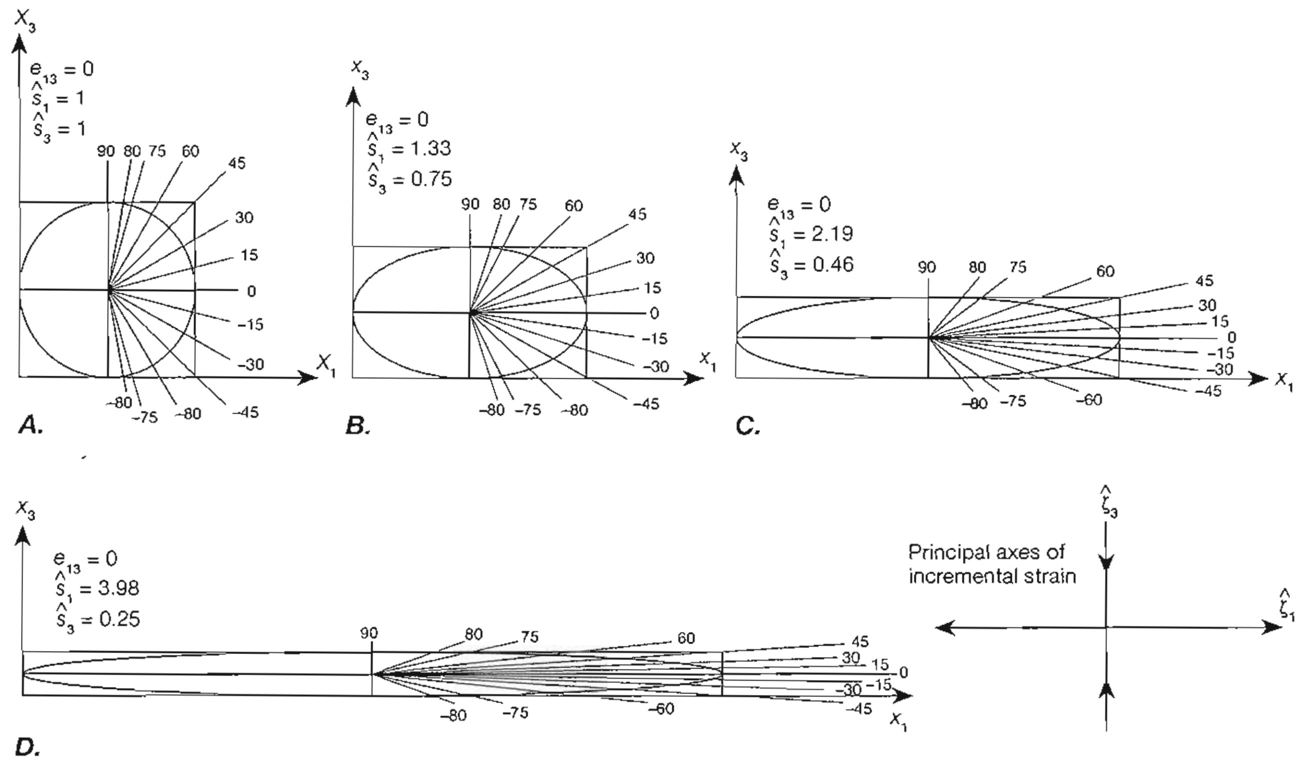


Figure 15.14 States of strain during a steady progressive pure shear. The axes at the bottom right of the figure indicate the constant orientation of the principal axes of incremental stretch. Material lines are labeled by the angle they make with X_1 in the undeformed state. The lines 0 and 90 are the only ones that do not rotate during the deformation, and they are always parallel to the principal axes of strain. The magnitudes of the principal srretches in each diagram are rhe same as for rhe corresponding diagram in Figure 15.15.

tative description of these motions.) For these examples, the incremental strain ellipse has the geometric properties of either pure shear (Figure 15.9) or simple shear (Figure 15.11) for each increment of strain through time.

Figures 15.14 and 15.15 illustrate the consequences of progressive pure shear and progressive simple shear, respectively. Part *A* in each figure is the undeformed state, showing a sheaf of material lines. In Figure 15.14*A*, the material lines are oriented at regular angular intervals, and each line is labeled with the angle it originally makes with the X_1 axis. In Figure 15.15*A*, the material lines are parallel to the axes of inverse strain for the state of strain in the diagram labeled with the corresponding letter. For example, the material lines C and C′ in part *A*, are parallel to the principal axes of inverse strain for the strain state shown in part C. These lines are rotated by the deformation into the orientations shown by *c* and *c*′, which become parallel to the principal axes of strain in part C. Parts *B* through *D* in both figures show the evolution of both the strain ellipse and the orientations of the same material lines as appear in part *A*. The corresponding diagrams in the two figures show the same states of strain, although the orientations of the principal axes are different (see Figure 15.12).

A comparison of Figures 15.14 and 15.15 shows the following significant differences in behavior:

1. With respect to the coordinate axes, the principal axes of strain do not rotate in progressive pure shear, but in progressive simple shear they do. Thus the former is an **irrotational**, and the latter a **rotational** progressive deformation. The difference in behavior of the principal strain axes is described by the **vorticity** of the deformation,[9] which is a measure of the average rate of rotation of material lines of all orientations about each coordinate axis.

   The vorticity is zero for irrotational deformations and nonzero for rotational deformations. In Figure 15.14, for example, the material lines in the upper-right quadrant rotate in the opposite sense to those in the lower right.

[9] Technically, the vorticity vector ω is the curl of the velocity ($\omega = \nabla \times V$), which has the three components

$$[\omega_1, \omega_2, \omega_3] \equiv [(\partial v_3/\partial x_2 - \partial v_2/\partial x_3), (\partial v_1/\partial x_3 - \partial v_3/\partial x_1), (\partial v_2/\partial x_1 - \partial v_1/\partial x_2)]$$

It is related to the spin tensor, which is the antisymmetric part of the velocity gradient tensor (see footnote in Section 19.7).

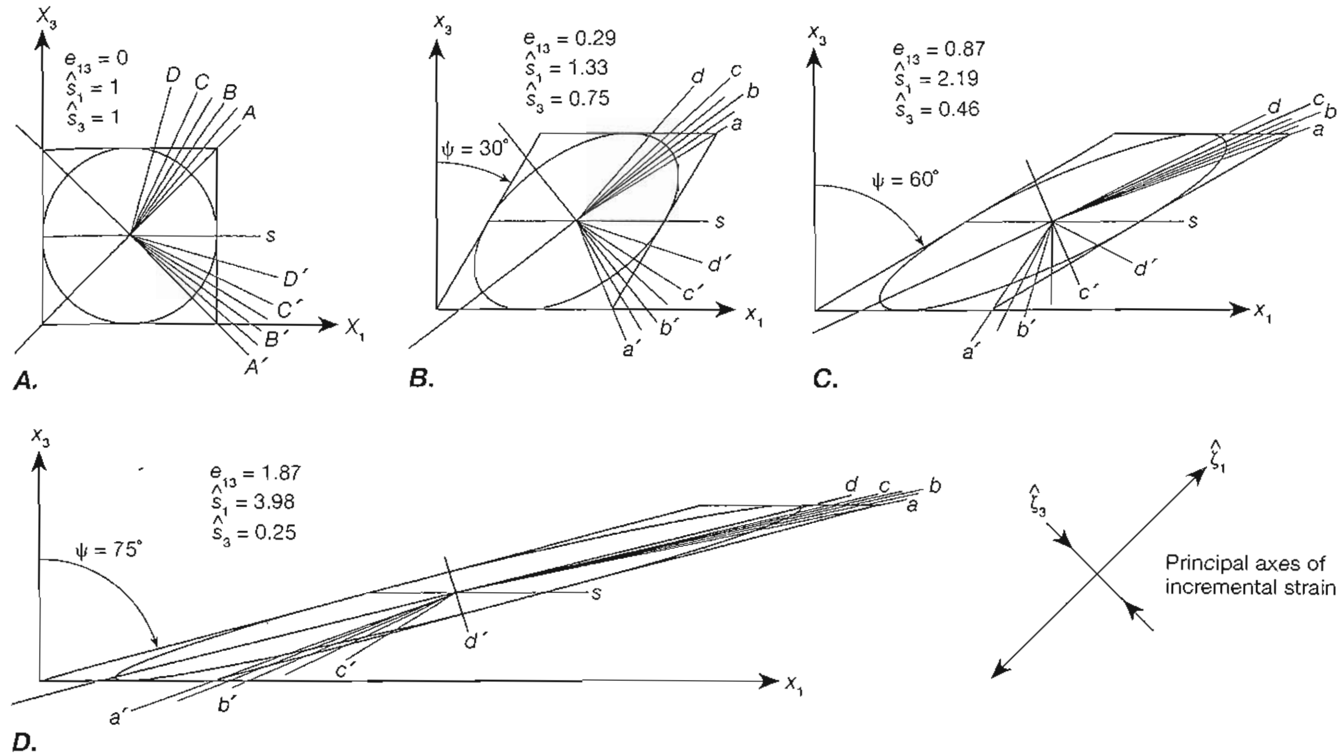


Figure 15.15 States of strain during steady progressive simple shear. The axes at the bottom right of the figure indicate the constant orientation of the principal axes of incremental stretch. The pairs of material lines in the undeformed state labeled (*B* and *B′*), (*C* and *C′*), and (*D* and *D′*) are parallel to the principal axes of inverse strain for the strain states shown in parts *B*, *C*, and *D*, respectively. These pairs of material lines take on the orientations in the deformed stares indicated by the lines labeled in the equivalent lower-ease letters, and each pair becomes parallel to the principal axes of strain in the diagram labeled with the same letter as the line pair. Thus the material lines rotate past the prineipal axes of srrain, which themselves are not marerial lines. *S* and *s* indicate a marerial line parallel to the shear plane. This is the only orienration of line for which rhe orienrarion and length are constant throughout the deformarion.

Because material lines oriented symmetrically relative to the x_1 axis have exactly opposite rates of rotation, the average over all orientations must be zero. In contrast, all the material lines in Figure 15.15 rotate in the same sense, so the average rate of rotation is nonzero.

2. In progressive pure shear, the principal axes of finite strain are always parallel to, or coaxial with, the principal axes of incremental strain. The deformation is therefore a coaxial progressive deformation. In progressive simple shear, the principal axes of finite strain rotate with respect to those of incremental strain, and this characreristic defines a noncoaxial progressive deformation. Note that for progressive simple shear the principal axes of incremental strain are always at a 45° angle to the shear plane.

   The terms *irrotational* and *coaxial* are not synonymous, nor are *rotational* and *noncoaxial*. The difference is in the reference frame from which the rotation is determined. A deformation is totational or irrotational depending on how the principal axes of finite strain behave with respect to the coordinate system, which is always somewhat arbitrarily defined by the observer. A deformation is coaxial or noncoaxial depending on how the principal axes of finite strain behave with respect to the principal axes of incremental strain. This reference frame is intrinsic to the geometry of the deformation itself and is therefore not arbitrary. Thus the descriprion of a progressive deformation as coaxial or noncoaxial is somewhat more fundamental than the description as rotational or irrotational, especially in geologic situations in which the best choice of an external coordinate system is not obvious.

3. In progressive pure shear, all material lines rotate during the deformation except rhose parallel to the principal axes of strain. The lines rotate toward parallelism with the $\hat{s}_1$ direction.

Note that the term *irrotational* refers only to the behavior of the principal axes of strain and to the average motion of all material lines, nor to the motion of a specific material line. In progressive simple shear, all lines except those parallel to the shear plane rotate during the deformation, and the rotation rate of any line decreases with decreasing angle between the line and the shear plane.

4. The lines that rotate most rapidly in progressive pure shear are those at an angle of 45° to the principal axes of the incremental strain ellipse. In progressive simple shear, the lines that rotate most rapidly are normal to the shear plane, and these lines are also at a 45° angle from the principal axes of incremental strain. Lines parallel to the shear plane, however, do not rotate at all, and they too are 45° from the principal axes of incremental strain.
5. In progressive pure shear, the same pair of material lines remains parallel to the principal axes of strain throughout the deformation. In progressive simple shear, material lines rotate through the principal axes of strain. This characteristic shows that the principal axes of strain are not in general material lines. During progressive simple shear material lines that are parallel to the principal axes at any time were originally orthogonal in the undeformed state. During the deformation, however, any such pair of lines is sheared out of orthogonality, then back into orthogonality when they are parallel to the principal axes, and finally out of orthogonality again (lines C and C′ in Figure 15.15).
6. In both progressive pure and progressive simple shear, the stretch of material lines depends on their orientation. Some lines experience a history only of shortening, others experience only lengthening, and still others experience initial shortening followed by lengthening and can end up being either shorter or longer than they were originally. The patrern of variation determines what types of structures can develop. We discuss this furrher in the next section.

If the deformation stops at any time, the final state of strain can always be related to the initial state in Figure 15.14 by a pure shear strain or in Figure 15.15 by a simple shear strain. The converse of this statement, however, is not true: If a final srate of strain can be relared to the inirial state either by a pure shear strain or by a simple shear strain, it does not follow that the final state of strain was the result of a progressive pure shear or a progressive simple shear, respectively. There are an infinite number of srrain paths thar lead from an undeformed state to a deformed state, and the final state of strain does not by itself provide sufficient information for any of the paths to be distinguished. It is very important ro remember this when interpreting the strain in rocks.

From the foregoing discussion, it is evident that if a progressive deformatiou is noncoaxial, the principal axes of finite strain rotate relative to those of incremental strain, and that the principal axes of incremental strain are constant in orientation only if the deformation is steady. It should not be surprising, therefore, that the principal axes of finite strain are not in general parallel to the principal axes of stress. In fact, we see in Chapter 18, where we discuss the relationships between stress and strain, that for steady morions of homogeneous isotropic materials, the prineipal stress axes are parallel to the principal axes of *incremental* strain or of strain rate. Because most natural deformations are probably not steady, even this relationship may not be accutate for interpreting deformation that we observe in rocks. Thus as a general rule, structures should always be interpreted in terms of the principal axes of strain. Only under very special circumstances can useful inferences be made about the orientations of the principal stress axes.

15.5 Progressive Stretch of Material Lines

If the unit circle is superposed on the finite strain ellipse, the radii to the intersection points define lines of no finite extension ($e_n = 0$), which are lines that are the same length as they were in rhe undeformed state ($s_n = 1$). These lines divide the ellipse (Figure 15.16*A*) into sectors in which radii are longer than they were originally ($s_n > 1$, labeled L) and sectors in which the radii are shorter ($0 < s_n < 1$; labeled S).

We can also examine a similar superposition of the unit circle on the incremental strain ellipse. For generality, we show in Figure 15.16*A* and *B* the finite and incremental principal strains in a relative orientation rhat can occur in nature only if the incremental principal axes have changed orientation during the deformation. The intersection of the circle wirh the incremental ellipse defines a pair of lines rhat instanraneously are not changing length. These lines divide the incremental strain ellipse (Figure 15.16*B*) into sectors in which lines are becoming longer ($[ds_n/dt] > 0$; labeled $\dot{L}$) and sectors in which the lines are becoming shorrer ($[ds_n/dt] < 0$; labeled $\dot{S}$).

The sector boundaries on the ineremental strain ellipse (Figure 15.16*B*) are not in the same orientation as those on the finite strain ellipse (Figure 15.16*A*), and

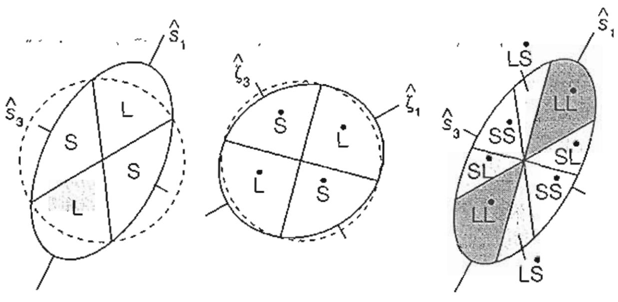
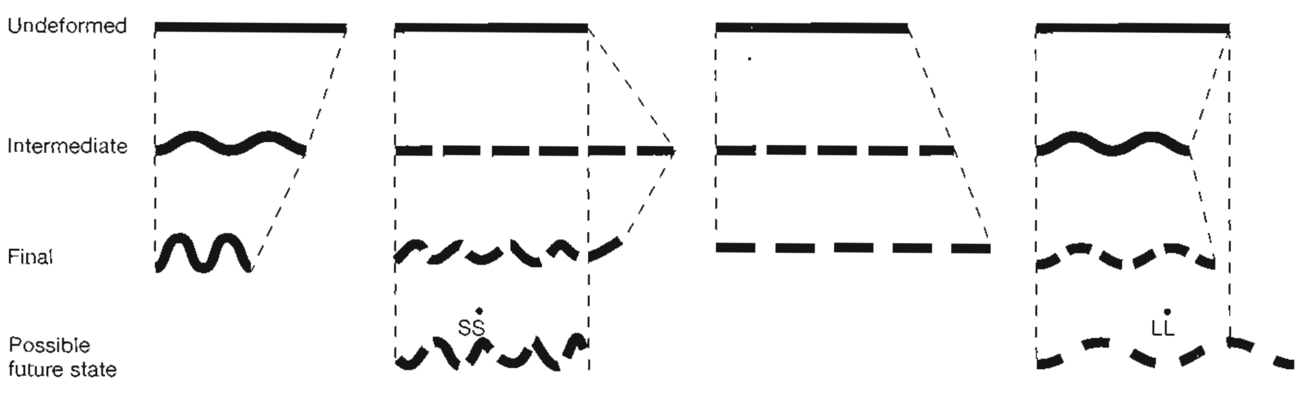


Figure 15.16 Geometry of finite and incremental strain ellipses for a deformation in which the incremental strain is superposed on a preexisting homogeneous strain. For generality, we have chosen an orientation of the finite strain ellipse that can be formed only from an unsteady deformation, characterized by an incremental strain ellipse whose principal axes change orientation during the deformation. *A.* The strain ellipse, showing lines of no finite extension that define sectors in which radial material lines have been lengthened (L) or shortened (S) by the deformation. The unit circle is shown dashed. *B.* The incremental strain ellipse, showing lines of no rate of extension that divide the ellipse into sectors in which radial material lines are being lengthened (L̇) (positive rate of change of stretch) and sectors in which radial material lines are being shortened (Ṡ) (negative rate of change of stretch). *C.* The combination of the two sets of sectors from parts *A* and *B* on the strain ellipse defines sectors in which radial material lines have different combinations of stretch and rate of stretch.

because material lines in general rotate during a deformation, they can pass from one sector into another. Thus the finite strain ellipse can be divided into sectors in each of which the material lines have a different history of stretching (Figure 15.16C). The different possible histories are illustrated in Figure 15.17, where shortening of material lines is represented as folding or imbrication, and lengthening of material lines is represented as boudinage. In sectors labeled SṠ, lines are shorter than the original length and have a history of continuous shortening (Figure 15.17*A*). In sectors labeled LṠ, lines are longer than the original length indicating an initial history of lengthening, but they are now shortening (Figure 15.17*B*); with continued deformation they may end up shorter than their initial length and therefore positioned in the SṠ sector (see Figure 15.17*E*). In sectors LL̇, lines are longer and have a history of continuous lengthening (Figure 15.17C); and in sectors SL̇, lines are shorter, indicating an initial history of shortening, but they are now lengthening (Figure 15.17*D*). With continued deformation they may end up longer than their initial length and therefore in the (LL̇) sector. Thus (SṠ) sectors may be subdivided according to whether or not the lines had an initial history of lengthening (compare Figure 15.17*A*, *B*). Similarly, (LL̇) sectors may be subdivided according to whether or not



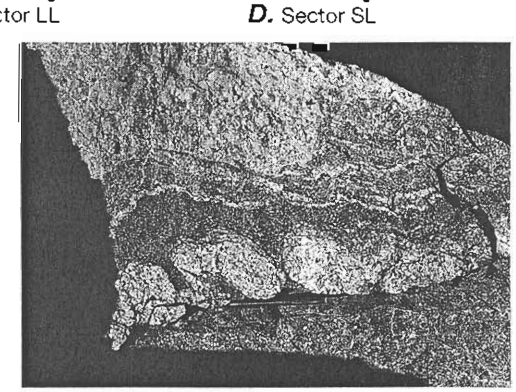


Figure 15.17 The histories of progressive deformation for competent layers oriented within the different sectors shown in Figure 15.16C. The undeformed, intermediate, and final states are points along the deformation path. *A.* Sectors SṠ: shorter and being shortened. The layer is continuously folded. *B.* Sectors LṠ: longer and being shortened. The layer was initially boudined and subsequently shortened, which caused folding and imbrication of the boudins. The "final" overall length is greater than the initial length, but continued shortening could make it less, thereby transferring the line into the SṠ sector. *C.* Sectors LL̇: longer and being lengthened. The layer is continuously boudinaged. *D.* Sectors SL̇: shorter and being lengthened. The layer is initially folded and subsequently boudinaged. The "final" overall length is smaller than the original length, but continued lengthening could make it longer, thereby transferring the line into the LL̇ sector. *E.* Boudins that have been shortened after formation, illustrating the deformational history in part *B*.

the lines had an initial history of shortening (compare Figures 15.17*C*, *D*: see lighter grey portions of $L\dot{L}$ sectors in Figure 15.18*A*, *B*).

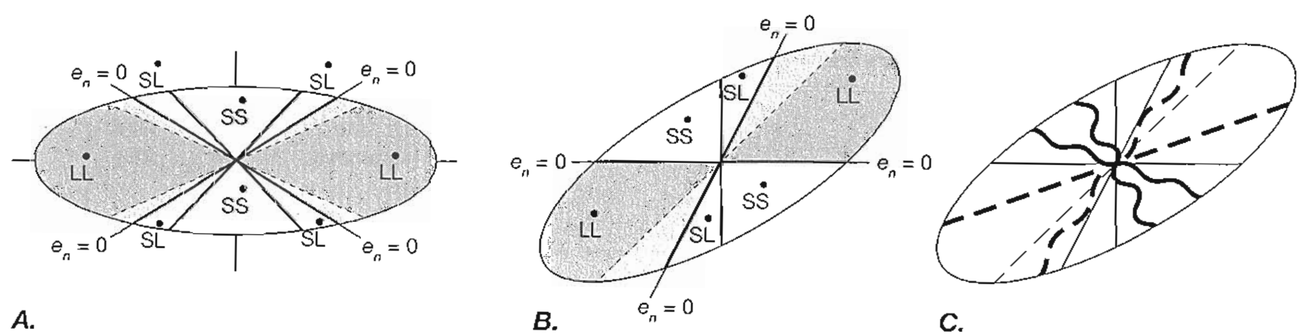
Thus, depending on the orientation of the material line with respect to the strain axes, the same deformation can produce folds, boudinage, boudinaged folds, or folded and imbricated boudins. The distribution of such sectors for progressive pure shear and for progressive simple shear is shown in Figure 15.18*A*, *B*, respectively. The main difference in the distribution of sectors about the principal axes of strain is the absence of an ($S\dot{L}$) sector for progressive simple shear subparallel to the shear plane. Thus the sectors of the strain ellipse for progressive pure shear have an overall orthorhombic symmetry, whereas the sectors for progressive simple shear have an overall monoclinic symmetry. When these aspects of the deformation are taken into account, an arbitrary deformation cannot be reproduced by the sequence of operations indicated in Figure 15.12 because, for example, the rotation of the sectors with the strain ellipse produced by progressive pure shear (Figure 15.18*A*) does not reproduce the sectors in the strain ellipse formed by progressive simple shear (Figure 15.18*B*), even though the strain ellipses themselves are the same shape.

In principle, then, it should be possible to distinguish some features of the strain history, such as coaxial and noncoaxial progressive deformations, by examining the relationship between the deformational structures in the rock and their orientations. For example, if veins are intruded into a rock in a variety of orientations, subsequent deformation could cause veins to form folds and/or boudins depending on their orientation relative to the principal stretches. The observed distribution of these structures defines the sectors of the finite strain ellipse (Figure 15.18*C*). In practice, however, the sector patterns are difficult to establish. The distribution of orientations of deformed layers is usually not ideal (Figure 15.18*D*), and layers can shorten and thicken without folding or can lengthen and thin without boudinage. Despite its limited practical application, this analysis demonstrates the important fact that no single type of structure is uniquely indicative of a particular geometry of deformation.

15.6 The Representation of Strain States and Strain Histories

It is often useful to compare various states of strain in order to show, for example, how they are related to one another in heterogeneously deformed rocks or to illustrate the sequence of strain states that represents a particular progressive deformation. Such a comparison is easily made by plotting the information on a **Flinn diagram**, on which the ordinate and abcissa are the ratios *a* and *b* of the principal stretches, defined by

$$a = \frac{\hat{s}_1}{\hat{s}_2} \qquad b = \frac{\hat{s}_2}{\hat{s}_3} \tag{15.21}$$



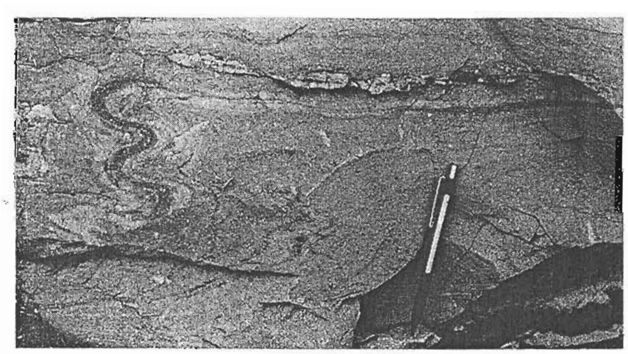


Figure 15.18 Distribution of sectors of stretch and stretching. Material lines in the lighter grey parts of the $L\dot{L}$ sectors have an initial history of shortening followed by lengthening (see bottom of Figure 15.17*D*). *A*. Progressive pure shear. *B*. Progressive simple shear. This case differs from progressive pure shear mainly in the lack of symmetry of the ($S\dot{L}$) sectors about the principal axes of strain. C. Structures developed in competent layers in an incompetent matrix consistent with the sectors for progressive simple shear. *D*. Folding of a layer (left) and simultaneous boudinage of a perpendicular layer (horizontal above pencil) during deformation of a marble.

The study of geologic strains rarely includes the volumetric strain, because it is very uncommon to know the original *size* of a strained object such as a fossil, even though its original *shape* may be known. Thus we can frequently determine the relative lengths of the principal axes of the strain ellipsoid but not the absolute lengths. Because the Flinn diagram is a plot of the ratios of the principal stretches, it can be used to show the shape of a strain ellipsoid, but not the size.

The origin of the coordinate axes for the Flinn diagram is generally taken to be (1, 1) because a and b cannot be less than 1, as can be seen from the second Equation (15.14) and Equation (15.21). Any strain ellipsoid plots at a particular point on the Flinn diagram, and the slope k of the line from the origin (1, 1) to that point is

$$k = \frac{a-1}{b-1} = \frac{\hat{s}_1\hat{s}_3 - \hat{s}_2\hat{s}_3}{(\hat{s}_2)^2 - \hat{s}_2\hat{s}_3} \qquad (15.22)$$

The value of k provides a useful way of classifying the types of constant-volume ellipsoids (Figure 15.19). Three lines, for $k = 0$, $k = 1$, and $k = \infty$, divide the graph into two fields, with ellipsoids of different characteristics plotting along each line and within each field. The field of flattening strain comprises the region for which $0 \leq k < 1$. The line $k = 0$ characterizes oblate uniaxial ellipsoids (pancake-shaped; $\hat{s}_1 = \hat{s}_2 > 1 > \hat{s}_3$), and the range $0 < k < 1$ characterizes oblate triaxial ellipsoids ($\hat{s}_1 > \hat{s}_2 > \hat{s}_3$). The line $k = 1$ characterizes all plane strain ellipsoids ($\hat{s}_1 > \hat{s}_2 = 1 > \hat{s}_3$). The field of constrictional strain includes the values $1 < k \leq \infty$. The range $1 < k < \infty$ describes prolate triaxial ellipsoids ($\hat{s}_1 > 1 > \hat{s}_2 > \hat{s}_3$), and the line $k = \infty$ describes prolate uniaxial ellipsoids (cigar-shaped, $\hat{s}_1 > 1 > \hat{s}_2 = \hat{s}_3$). The values of the stretches given here apply only to constant-volume strains (Equations 15.18).

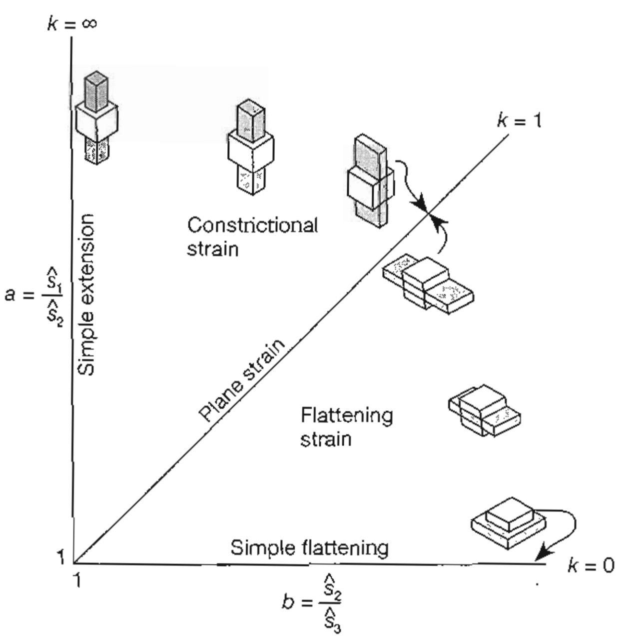


Figure 15.19 Flinn diagram showing the three lines ($k = 0$, $k = 1$, and $k = \infty$) and the two fields ($0 < k < 1$ and $1 < k < \infty$) of finite strain ellipsoids for constant-volume deformation.

The Flinn diagram lends itself well to the representation of strain paths, which define the sequence of strain states through which a body passes in a progressive deformation. Steady motions produce strain paths that plot as straight lines. In geologic deformation, however, steady motions over long periods of time are probably the exception, and curved paths, which may even cross from the constrictional field into the flattening field, or vice versa, are probably common. The diagram makes no distinction, however, between coaxial and noncoaxial progressive deformations. Progressive pure shear and progressive simple shear, for example, are both constant-volume progressive plane deformations that plot along the line $k = 1$. This fact shows that the rotational component of any deformation, which distinguishes pure shear from simple shear, for example, is not represented on a Flinn diagram.

Volumetric deformation is easy to represent on the Flinn diagram. Because plane strain geometry ($\hat{s}_2 = 1$) must always separate the field of constriction ($\hat{s}_2 < 1$) from the field of flattening ($\hat{s}_2 > 1$), the location of this boundary separates constrictive from flattening strains even when the volume is not constant. In order to determine the equation for the line of plane strain when the volume is not constant, we take $\hat{s}_2 = 1$ in Equation (15.21), which gives

$$a = \hat{s}_1 \qquad b = \frac{1}{\hat{s}_3} \qquad (15.23)$$

We can then express the equation for volumetric stretch in plane strain (the first Equation 15.17) in terms of a and b by using Equation (15.23).

$$s_v = a/b \qquad \text{or} \qquad a = s_v b \qquad (15.24)$$

The second Equation (15.24) is the equation for the plane strain line on the Flinn diagram in terms of the volumetric stretch. Taking the natural logarithm of both sides gives an alternative form:

$$\ln a = \ln s_v + \ln b \qquad (15.25)$$

The base-10 logarithm could also be used. The second Equation (15.24) shows that the volumetric stretch s_v determines the slope of the line through the point $(a, b) = (0,0)$ on the Flinn diagram that separates constrictional strain from flattening strain. Note that in general, these lines do not pass through the origin of the Flinn diagram $(a, b) = (1, 1)$. Only when the volume is constant ($s_v = 1$) is the slope of the plane strain line equal to 1, in which case $k = 1$ also, and the line passes through the origin of the Flinn diagram $(a, b) = (1, 1)$.

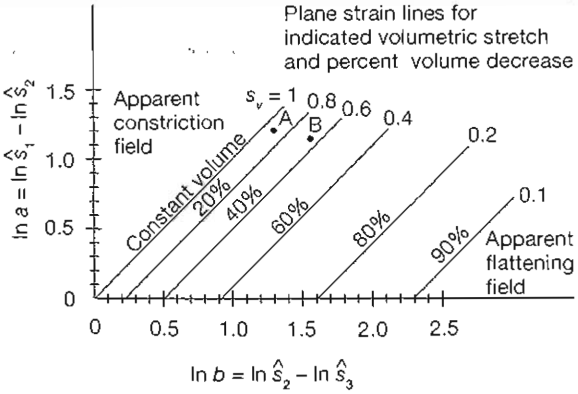


Figure 15.20 Logarithmic Flinn diagram showing the plane strain boundary lines (Equation 15.25) between the fields of flattening and constrictional strain for various amounts of volumetric stretch s_v (Equation 15.16).

The logarithmic Flinn diagram, on which the axes are ln *a* and ln *b*, is more conveuient for showing the effects of volumetric deformation (Figure 15.20). Equation (15.25) shows that on this form of the diagram, the plane strain line maintains a constant slope of 1, and the volumetric stretch determines the intercept. Each line on Figure 15.20 represents the plane strain line for a different volumetric stretch, as labeled, and each line therefore separates the field of constrictive strain above from the field of flattening strain below.

The danger of iuterpreting strain measuremeuts without knowing the volumetric stretch is evident from Figure 15.20. A strain ellipsoid that plots at point *A*, for example, would be in the flatteniug field for $s_v = 1$ but in the constrictive field for $s_v \leq 0.8$. Similarly, a strain ellipsoid that plots at point *B* would be in the flattening field for $1 \geq s_v \geq 0.8$ but iu the constrictive field for $s_v \leq 0.6$. Thus plotting strain ellipses on the Flinn diagram without knowing the volumetric stretch can be misleading, and the common assumption of constant-volume deformation for rocks can lead to incorrect interpretations.

15.7 Homogeneous and Inhomogeneous Deformation

So far in this chapter we have restricted our discussion to homogeneous strains. As we noted at the beginning of this chapter, if we are interested in the inhomogeneous distribution of strain, such as in the formation of a fold, we assume the deformed body can be divided into volumes that are sufficiently small for the deformation to be described as locally homogeneous. The variation of these local strains across the body describes the inhomogeneous strain distribution. For any real material, we must realize that the description of a deformation as homogeneous at any particular scale is the result of averaging the deformation over volumes that are large compared with the scale of inhomogeneities that are of no immediate interest, but small compared with the scale at which the inhomogeneous distribution of strain is of interest.

Figure 15.21, for example, shows the so-called deck-of-cards model for forming a passive shear fold (see also Figure 12.8). As discussed in Section 12.2, the deformation is accomplished by a discontinuity in the shear displacement at the card surfaces, with no deformation at all of the individual cards. On the scale of a fold limb, however, the deformation in this example can be regarded as homogeneous simple shear, and it produces the average strain ellipse shown on each fold limb in the figure. Thus the description of the strain as homogeneous results from averaging the strain over a region thar is large compared with the thickness of the cards, but small compared with the wavelength of the fold. In other words, the homogeneity depends on scale.

The variety of scales on which we could consider a deformation to be homogeneous is illustrated in Figure 15.22. In Figure 15.22*A*, the body of folded rock measures about 1 km in length. The scale of the whole block is large compared with the wavelength of the folds, but small compared with the dimension of a mountain belt. At this scale, the average deformation is homogeneous and is represented by the strain ellipse showu beside the block.

When we look at a scale eomparable to the fold wavelength, however, the strain is no longer homogeneous (Figure 15.22*B*). We then describe the deformation in terms of the variation in local strain, which is considered homogeneous on a scale, for example, of about

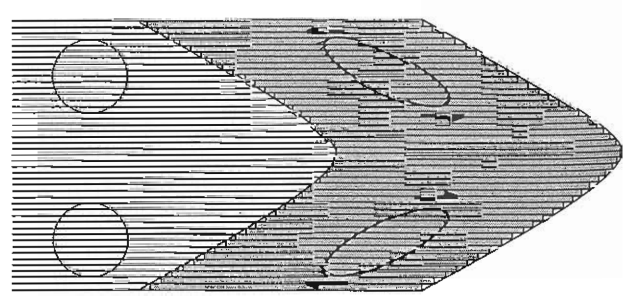


Figure 15.21 Deck-of-cards model of passive-shear folding. On each card, the arcs of the undeformed circle are displaced so as to approximate the shape of the strain ellipse.

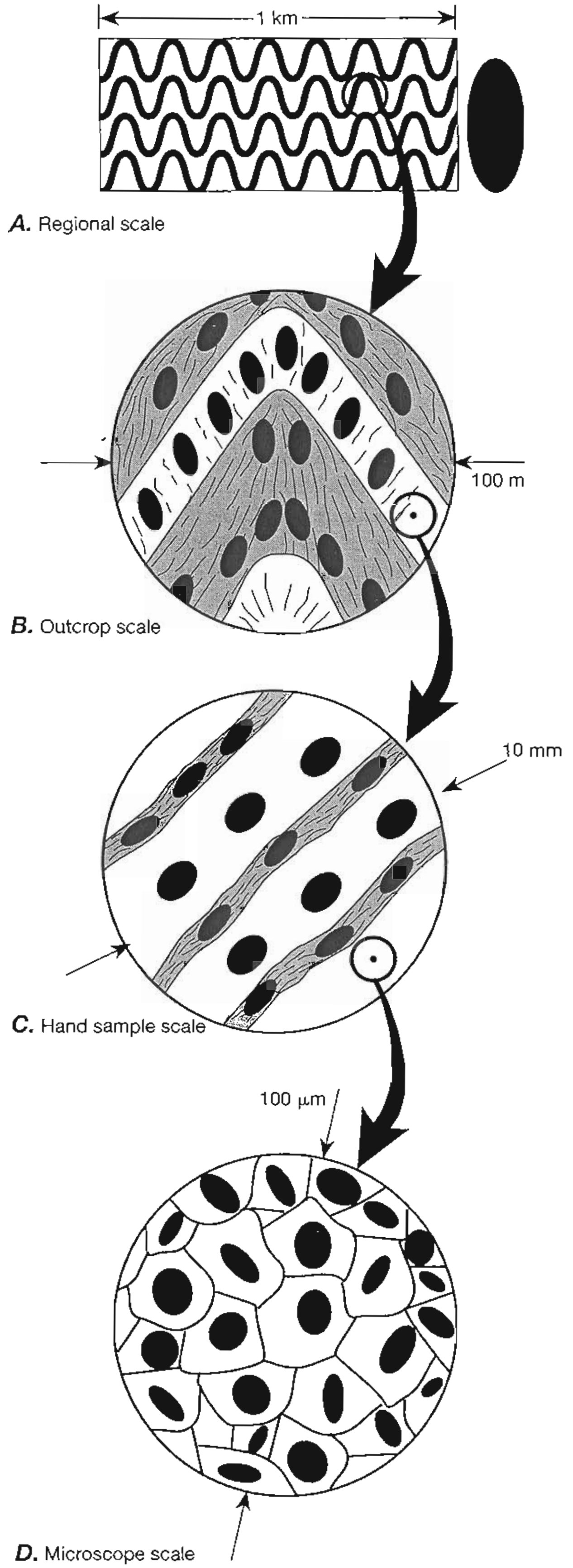


Figure 15.22 Scales of homogeneous and inhomogeneous strain. In each diagram, the volume over which the strain is averaged to form a locally homogeneous strain ellipse can be viewed at a smaller scale at which the strain distribution is inhomogeneous.

a meter. That scale is small compared with the wavelength of the fold, but large compared with the inhomogeneities in strain that might be present, for example, if the layer were a sandstone containing a spaced foliation.

When we shift scales again, down to the level of the spaced foliation (Figure 15.22C), we again find an inhomogeneous distribution of local strain. In this case, the local strain is averaged over a volume small relative to the spacing of the foliation domains, but large relative to the grain size.

Another shift in scale brings us down to the scale of the grains (Figure 15.22*D*), where the local strain is again inhomogeneous and the strain in each grain is averaged over a volume that is large compared with the scale of crystal lattice imperfections.

Thus we can consider the strain to be "homogeneous" on a scale that is small compared with the particular structure within which we want to determine the strain distribution, but large compared with the scale of inhomogeneities in which we are not interested and over which we want to average the deformation.

Additional Readings

Eringen, A. C. 1967. *Mechanics of continua.* New York: Wiley.

Flinn, D. 1962. On folding during three dimensional progressive deformation. *Q. J. Geol. Soc. Lond.* 118: 385–428.

Means, W. D. 1976. *Stress and strain: Basic concepts of continuum mechanics for geologists.* New York: Springer-Verlag.

Passchier, C. W. 1990. Reconstruction of deformation and flow parameters from deformed vein sets. *Tectonophysics* 180(2–4): 185–199.

Ramsay, J. G. 1967. *Folding and fracturing of rocks.* New York: McGraw-Hill.

Ramsay, J. G. 1976. Displacement and strain. *Phil. Trans. Roy. Soc. Lond.* A283: 3–25.

Ramsay, J. G., and M. I. Huber. 1983. *The techniques of modern structural geology.* Vol. 1: Strain analysis. New York: Academic Press.

Wood, D. S., and P. E. Holm. 1980. Quantitative analysis of strain heterogeneity as a function of temperature and strain rate. *Tectonophysics* 66: 1–14.