Math 398

Discrete Mathematics
for
Elementary Teachers

Winter 2011

Activities Course Pack

Kruczek

Discrete Mathematics

What is Discrete Mathematics? Below, you will find definitions from various sources (some reliable, some….not so much):

- Discrete mathematics is the study of mathematical structures that are fundamentally discrete in the sense of not supporting or requiring the notion of continuity. ... en.wikipedia.org/wiki/Discrete_mathematics

- The study of mathematical properties of sets and systems that have a countable number of elements.
  https://www.doe.mass.edu/frameworks/math/1996/gloss.html

- Discrete mathematics is the branch of mathematics dealing with objects that can assume only distinct, separated values. The term "discrete mathematics" is therefore used in contrast with "continuous mathematics," which is the branch of mathematics dealing with objects that can vary smoothly (and which includes, for example, calculus). Whereas discrete objects can often be characterized by integers, continuous objects require real numbers.

  The study of how discrete objects combine with one another and the probabilities of various outcomes is known as combinatorics. Other fields of mathematics that are considered to be part of discrete mathematics include graph theory.

  The study of topics in discrete mathematics usually includes the study of algorithms, their implementations, and efficiencies. Discrete mathematics is the mathematical language of computer science, and as such, its importance has increased dramatically in recent decades.
  http://mathworld.wolfram.com/DiscreteMathematics.html

- For me, discrete mathematics is about counting, playing games, coloring, and solving puzzles. – Klay Kruczek

I cannot emphasize the following statements enough: ``The solutions of discrete mathematics problems can often be obtained using ad hoc arguments, possibly coupled with use of general theory. **One cannot always fall back on application of formulas or known results"** To me, this means you have to play around sometimes. The answer is not always right in front of your face. I got into mathematics because I love solving puzzles. Discrete mathematics is that for me.

To solve a discrete mathematics problem, you may have to perform the following steps:

- Make sure you understand what the question is asking.
- Determine what AN example of an object satisfying the requirements of the problem might look like.
- Do some computations for small cases in order to develop an idea of what exactly is going on. (This is probably most important)
- Try to develop some sort of systematic approach when doing small examples.
- Use reasoning and possibly creativity to obtain a solution to the problem.
- **<u>Spend more than 30 minutes trying to solve a problem.</u>**

Pigeonhole Principle:

The first type of problem we will consider is the following, which you may have come across in your life. If you have 11 pairs of shoes in a big box, how many shoes do you have pull out before you are guaranteed to find a matching pair? Pretend you cannot see the shoes in the box because it is dark.

1. How many students would be required to place soda orders, one soda per student, in order to ensure that at least one of the six sodas (Birch Beer, Cola, Diet Cola, Ginger Ale, Root Beer, Sprite) would be ordered by at least two students?

2. The colors of skittles are: red, orange, yellow, green, and purple. How many skittles would you have to grab from a package to ensure that you have grabbed at least four of one color?

3. Nine classes are offered. Each person enrolls in exactly one class. How many students will have to enroll to ensure at least one class size is at least 7?

4. There are eight math focus classes offered at WOU. What would be the least number of people needed to sign up for classes to ensure that there are 10 people in at least one class?

5. How many students do you need in your classroom so that you are guaranteed that at least two students have the same birthday?

Pigeonhole Principle:

6. We have 10 boxes labeled 1 through 10 into which we place pennies. How many pennies are required to ensure that at least one box contains **at least as many** pennies as the label on the box?

7. If my CD collection consists of 5 albums by Katy Perry and 11 albums by Linkin Park, how many CDs must I select in order to get two by the **same** artist?

8. If my CD collection consists of 5 albums by Katy Perry and 11 albums by Linkin Park, how many CDs must I select in order to get two by **each** artist?

9. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket to guarantee that either there are at least eight apples or at least six bananas, or at least nine oranges?

10. Suppose there are 6 people in a room. Show that there are either three people who know each other or 3 people who don't know each other. Does this work with 5 people?

Gauss Summing

I am sure you have all heard the story about Carl Friedrich Gauss. His teacher got mad at the class and told them to add the numbers 1 to 100 and give him the answer by the end of the class. About 30 seconds later Gauss gave him the answer of 5050. How did he do it?

Definition: An **arithmetic *sequence*** is a set of numbers such that the gap between consecutive terms is constant.
(Examples: {7, 14, 21, …,637} and {1, 5, 9, 13, …., 25})

The sum of the numbers in any finite arithmetic sequence is:

Using the formula above, find the following sums (for #1 – 8).
.

1. Find the sum of the 1st 198 positive integers: 1 + 2 + 3 + ... + 198.

2. Find the sum of the positive even numbers up to 100: 2 + 4 + … + 100.

3. Find the sum of the positive odd numbers up to 63: 1 + 3 + 5 + … + 63.

4. Find the sum of the 1st 400 positive even numbers.

5. Find the sum of the 1st 125 positive odd numbers.

6. Find the sum of the multiples of 3 that are less than 300.

7. Find the following sum: 17 + 21 + 25 + … + 245

8. Find the following sum: 31 + 38 + …. + 703.

9. Find the sum of the multiples of 3 that are less than 300 and not divisible by 4.

10. Find the sum of the 1st 100 positive cubes ($1^3 + 2^3 + 3^3 + \ldots + 100^3$).
(Hint: do smaller sums first!)

Definition: A **geometric *sequence*** is a set of numbers such that the ratio between consecutive terms is constant.
(Examples: $\{1, 2, 4, 8, \ldots, 2^n\}$ and $\{1, 5, 25, 125, \ldots, 5^n\}$)

11. Find the following sums:
    a. $1 + 2 + 4 + 8$

    b. $1 + 2 + 4 + 8 + 16$

    c. $1 + 2 + 4 + 8 + 16 + 32$

    d. $1 + 2 + 4 + 8 + 16 + 32 + 64$

    e. $1 + 2 + 4 + 8 + 16 + 32 + 64 + \ldots + 2^n$

12. Find the following sums:
    a. $1 + 3 + 9 + 27$

    b. $1 + 3 + 9 + 27 + 81$

    c. $1 + 3 + 9 + 27 + 81 + 243$

    d. $1 + 3 + 9 + 27 + 81 + \ldots + 3^n$

13. Find the following sum (where r is any number):

    $1 + r + r^2 + r^3 + \ldots + r^n$.

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Folding Paper

1. Fold the 2” x 8” strip of paper in half (to make a 2”x4” piece of paper) and in half again (to make a 2”x2” piece of paper).

2. Visualize what the paper will look like unfolded and count in your head the number of rectangles (not necessarily of the same size) that there will be when you unfold the paper.

3. You and your group members should come to a consensus of how many rectangles there are.

4. How many rectangles do think you will find? _____________

5. Now unfold the paper and actually count the rectangles.

6. How many rectangles did you actually find? _________________

7. Label your unfolded paper as below. List all of the rectangles you found.

A	B	C	D

8. Repeat steps 1-3, except fold the paper in half three times.

9. How many rectangles do think you will find? _____________

10. Now unfold the paper and actually count the rectangles.

11. How many rectangles did you actually find? _________________

12. Label your unfolded paper as below. Find a compact way to count the number of rectangles you found.

A	B	C	D	E	F	G	H

13. Repeat steps 1-3, except fold the paper in half four times.

14. How many rectangles do think you will find? _____________

15. How many did you actually find? ______________ (There is no need to identify the rectangles by letters, but you should be able to say how many of each "type" of rectangle there are.)

16. If you folded the paper in half N times, how many rectangles do you think you would have when you unfolded the paper? Explain your reasoning.

Set Theory, Venn Diagrams, and the Principle of Inclusion-Exclusion

Set Theory Refresher:

- A **set** is a collection of objects, called **elements**.

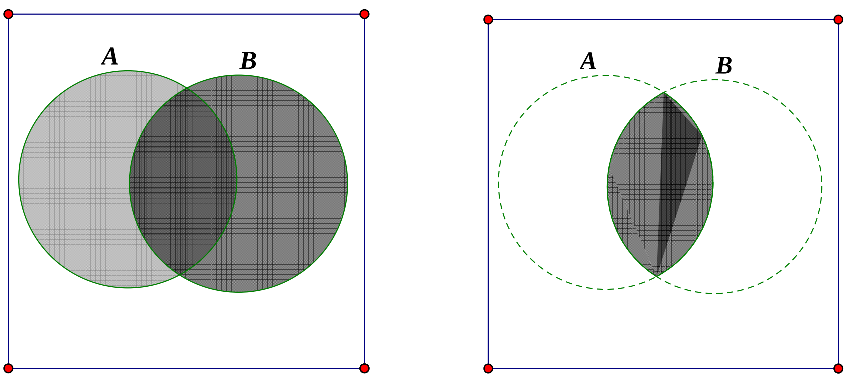
- To denote a set, we write, for example:
  A = {single digit non-negative even numbers} or A = {0, 2, 4, 6, 8}.

- The number of elements in A is called the **size of A** and is denoted **|A|.**
  e.g. For set A above, we have |A| = ________

- The elements of set A are 0, 2, 4, 6, and 8. Since 2 is an element of A, we write $2 \in A$.

- The **intersection** of two sets A and B, denoted **A ∩ B**, is the elements A and B have in common.

  The **union** of two sets A and B, denoted **A ∪ B**, is the set of elements in either A or B (or both).



- For example:
  A = {0, 2, 4, 6, 8} and B = {3, 6, 9, 12} means

  A ∩ B =   A ∪ B =

- We say C is a **subset** of a set A if everything in C is also in A. We denote this relationship as $C \subseteq A$.
  Example: C = {0,4,6} is a subset of A = {0,2,4,6,8}.

- The **complement** of a set B, denoted B' or B^c, is the set of elements in the "whole set", not in B.
  Example: if the whole set is W = {1, 2, 3, 4, 5, 6} and B = {3, 5}, then

  B' =

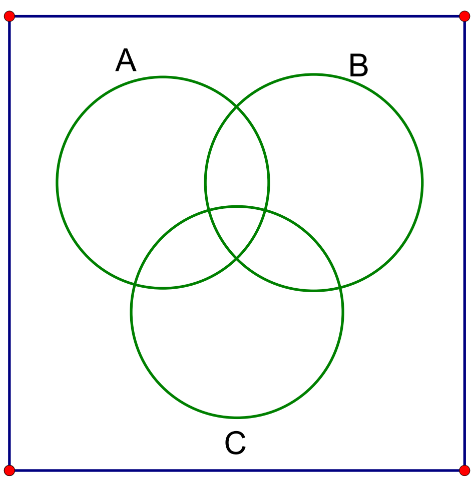
Venn Diagrams:

Sometimes, we are asked to determine a relationship among a finite number of sets. We use Venn diagrams to help us with these. Venn diagrams (for two to three sets) are usually made up of overlapping circles. Elements are put into the appropriate regions along the way. We will shade in the three-circle Venn diagrams below. Some people like to use the following techniques to help with shading:

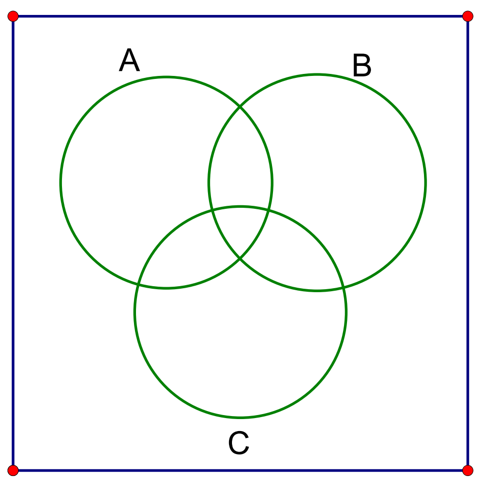
- Shade in the circle if there is no complement sign attached to the set;
- Shade in everything outside the circle if there is a complement sign attached to the set.
- A union sign indicates everything shaded at least once is in the union.
- An intersection sign indicates everything shaded twice (for two sets; and thrice for three sets) is in the intersection.
- If you have parentheses, reshade the union/intersection of the two/three sets inside the parentheses before considering sets outside the parentheses.
- shade the region represented by the set.

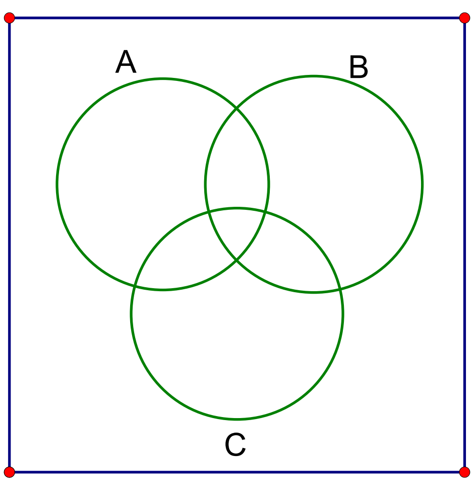
Shade in each of the following regions, represented by the set:

$(A \cup B) \cap C$

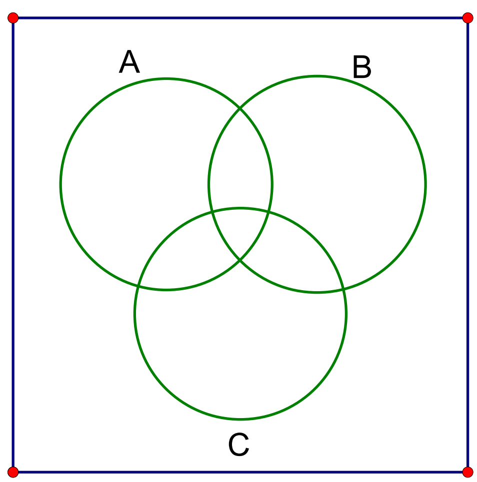


$(B \cup C) \cap A'$





$(A \cap C') \cup B'$



$(B \cap C)' \cap A'$

Venn Diagrams and the Principle of Inclusion - Exclusion:

- Use the Venn Diagrams below to help answer the questions.
- Take 4 blue 1” square tiles, 6 yellow 1” square tiles, 5 yellow blocks and all hexagons from the attribute blocks.

1. Label the circles of the 2-circle Venn diagram “A=Yellow” and “B=Hexagons.” Put the pieces in the appropriate spot in the Venn diagram.
   a. How many yellow pieces do you have?

   b. How many hexagons do you have?

   c. How many yellow hexagons do you have?

   d. How many total pieces do you have inside the two circles of your Venn diagram?

2. Label the circles of a 2-circle Venn diagram “A = Blue” and “B = Squares.” Put the pieces in the appropriate spot in the Venn diagram.
   a. How many blue pieces do you have?

   b. How many squares do you have?

   c. How many blue squares do you have?

   d. How many total pieces do you have inside the two circles of your Venn diagram?

3. If you have a set A with $|A|$ elements and a set B with $|B|$ elements, what is the size of the union, $|A \cup B|$, in relation to $|A|$, $|B|$ and $|A \cap B|$? Explain why your answer always works.

   $|A \cup B| =$

4. Label the circles of the 3-circle Venn diagram "A = Yellow", "B = Hexagons" and "C = .5cm thick." Appropriately place the pieces in the Venn diagram.

   a. How many yellow pieces do you have?

   b. How many .5cm thick pieces do you have?

   c. How many hexagons do you have?

   d. How many yellow hexagons do you have?

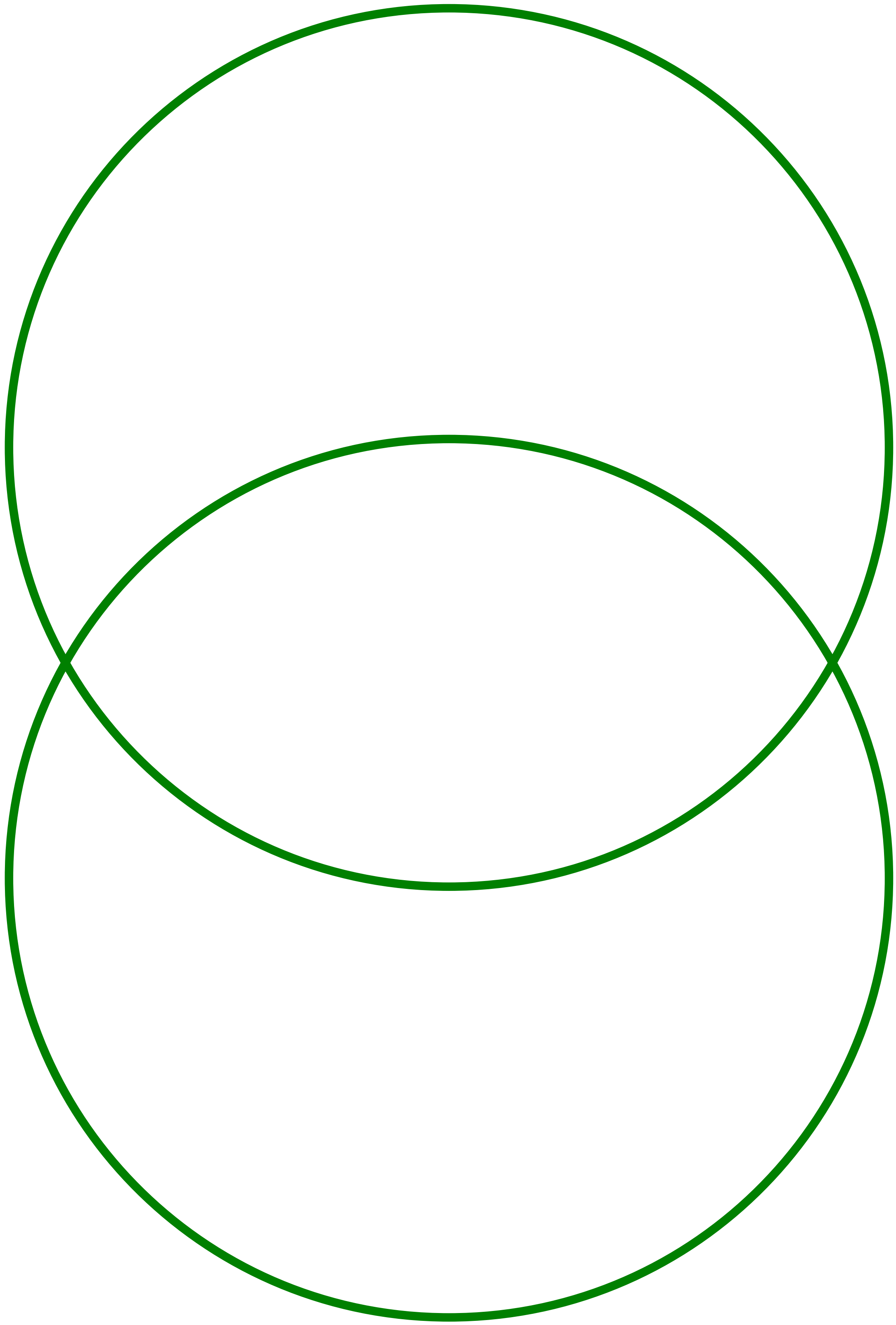
   e. How many .5cm thick hexagons do you have?

   f. How many .5cm thick yellow pieces do you have?

   g. How many .5cm thick yellow hexagons do you have?

   h. How many total pieces do you have, i.e., what is $|A \cup B \cup C|$?

   i. If you have a set A with $|A|$ elements, a set B with $|B|$ elements, and a set C with $|C|$ elements, then what is $|A \cup B \cup C|$, in relation to: $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$, and $|A \cap B \cap C|$? Explain your reasoning.



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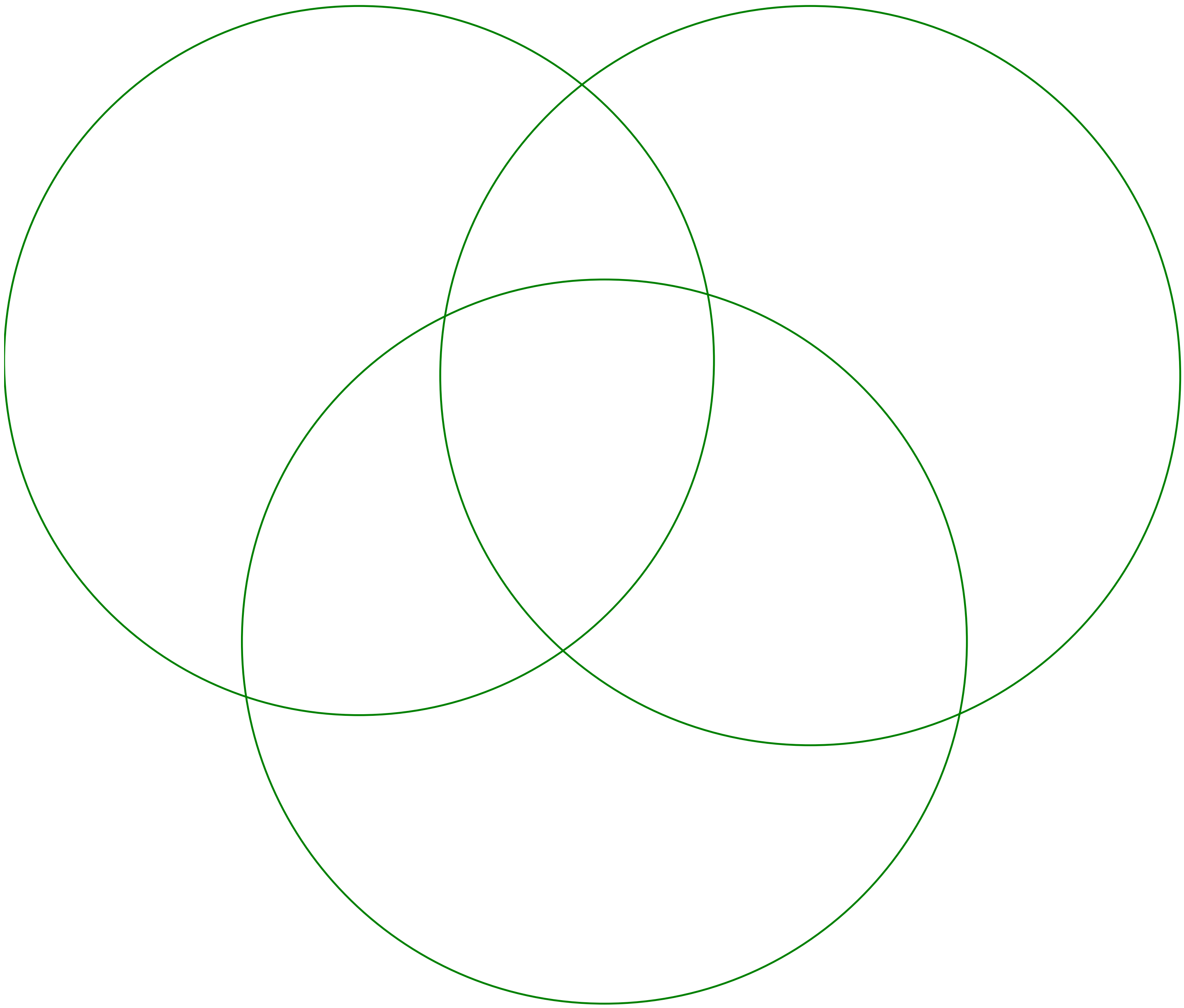

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Principle of Inclusion-Exclusion continued:
Suppose we have a set of objects S and a set of properties $P_1, P_2, \ldots, P_m$. For $k = 1, 2, ..., m$, let A_k be the elements in S with property k.

Example 1: Suppose P_6 means 6 divides the number. If $S = \{1, 2, \ldots, 100\}$, then
$A_6 =$

$|A_6| =$

Example 2: In our previous work, if P_1 = Hexagonal and S = {all attribute blocks}, then
$A_1 =$

$|A_1| =$

Principle of Inclusion-Exclusion: Suppose we have a set of objects S and a set of properties $P_1, P_2, \ldots, P_m$. For $k = 1, 2, ..., m$, let A_k be the elements in S with property k. The number of objects in S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by the formula:

$|A_1 \cup A_2 \cup \cdots \cup A_m| =$

More Problems:

5. Suppose that in a group of 18 job applicants, 10 have computer programming expertise, 5 have statistical expertise, and 2 have both programming and statistical expertise. How many members of the group have neither expertise?

6. How many integers between 1 and 60 (inclusive) are not divisible by 2, 3, or 5?

7. How many integers between 1 and 723 (inclusive) are not divisible by 2, 3, or 5?

8. How many integers between 0 and 99,999 (inclusive) have among their digits each of 2, 5, and 8?

9. **(Extra Credit on Homework)** Imagine that n women attend class and check their cell phones at the door because their professor does not like when they use them in class. The professor is a little absent-minded (as usual) and returns the cell phones at the end of class randomly. What is the probability that no woman receives her own cell phone back? Determine this answer for $n = 2, 3, 4, 5$.

Principles of Counting

When appropriate, use two different methods to answer the following questions:

1. Suppose you have 4 colors of construction paper: Red, Blue, Yellow and Green. You want to draw a picture on one piece of construction paper and frame the picture with another piece of construction paper.
   a. How many different frame/picture color combinations can you have if the frame and picture **can be** on the color of construction paper?
   b. How many different frame/picture color combinations can you have if the frame and picture have to be on different colors of construction paper?

2. Suppose you roll a regular 6-sided die and a 4-sided die simultaneously. How many different combinations can you have? (What does this mean exactly?)

3. In the game "Difference," you roll a 6-sided die, and your friend rolls a 4-sided die. You then subtract the smaller roll from the bigger roll. If the difference is 0 or 1, then you win $1. Otherwise, your friend wins $1. Is this a fair game for both you and your friend? What does a "fair game" mean?

4. Suppose that there are three major routes from Washington DC to Chicago, and five from Chicago to Los Angeles. How many routes are there from DC to LA that go through Chicago?

5. A **single-elimination tournament** is a type of tournament where the loser of each match is immediately eliminated from winning the championship or first prize in the event.
    a. If you want a single elimination basketball tournament to have 3 rounds, then how many teams should be in the tournament?
    b. If you wanted 5 rounds, how many teams should be in the tournament?
    c. If you wanted N rounds, how many teams should be in the tournament?

6. The NCAA Women's Basketball Tournament is a single elimination tournament with 64 teams in it. How many games need to be played to determine the national champion in Women's basketball?

7. If you have a single elimination tournament with N teams, how many games need to be played to determine the winner? Why does this make sense? (Try this for small examples.)

8. The NCAA Women's Basketball Tournament is a single elimination tournament with 64 teams in it. A betting service offers a deal, where if you bet $1 and pick the entire tournament right, then you win $100,000,000. Is this a good idea for them? Why or why not? (It's a good idea if the probability of winning is less than 1 in 100,000,000.)

9. Sunny Garden restaurant has 5 types of Beef Dishes and 8 types of Poultry Dishes.
    a. How many choices do you have if you get to choose one dish?
    b. How many choices do you have if you choose 2 different dishes?
    c. How many choices do you have if you choose one Beef Dish and one Poultry Dish?

10. In 1991, there were 9 teams in the Atlantic Coast Conference (ACC) {Maryland, UVA, NC St, Duke, UNC, Wake Forest, Clemson, Georgia Tech, Florida State}. There were also 8 teams in the Ivy League {Harvard, Dartmouth, Cornell, Columbia, Princeton, Penn, Yale, Brown}.

    a. The play-in round of the ACC basketball tournament had the two teams with the worst records play each other. How many possible match-ups are there for this game?

    b. In how many ways can you choose the ACC champion and the ACC runner-up? In how many ways, can you choose who finishes first, second and third in the conference?

    c. In the East Coast Bowl, the conference champion of the Ivy League and the conference champion of the ACC play each other. How many different match-ups are possible for this bowl?

    d. How many possible winners do you have for the East Coast Bowl?

11. **Generalization**: (**Counting Principles**)
    Suppose you have N_1 objects in set S_1, N_2 objects in set S_2 and N_3 objects in set S_3.

    a. How many different ways are there to choose one object from either S_1, S_2 or S_3?

    b. How many different ways are there to choose one object from S_1 and one from S_2 and one from S_3?

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Beginning Sample Problems for Combinations and Permutations

1. Three people, Smith, Jones and Brown are scheduled for job interviews. List the different orders that they can be interviewed. Does the order of how the interviews are performed matter here?

2. How many ways are there to choose which 3 books to buy from the group of 5 (Anna Karenina, Black Beauty, The Crucible, Don Quixote, Ethan Frome)? Does the order of how the books are chosen matter here?

3. You are trying different cheesecakes at the Cheesecake Factory. You want to try the following 6 cheesecakes: New York, Key Lime, Chocolate, Strawberry, Oreo, and Tiramisu. How many possible orders can you try them in? Does the order in which the cheesecakes are chosen matter here?

4. Suppose you have 7 students waiting on the sidelines to play a pick-up game of 2-on-2 basketball. How many ways can you select two of the players to form one team? Does the order of how we choose the players matter here?

5. Suppose the students then find out that the gym has decided to switch to 5-on-5 basketball. How many ways can you select 5 of these players to form a team (who plays which position does not matter)? Does the order of how we choose the players matter here?

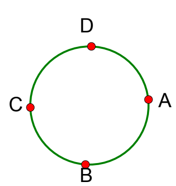
6. Is there anything similar in your answers to the previous two problems? Why do you think this happened?

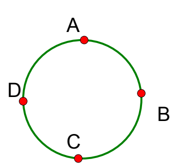
7. If a postal code consists of 5 digits, with no repeated digit, how many different postal codes are possible? Does the order of the digits matter here?

8. How many 4-letter “words” are there without any repeated letters? Does the order of the letters matter here?

9. Ten of your students get sent down to the principal’s office and are told to sit outside her office. Outside her office, there is a giant bench along the wall. How many different ways can she arrange the students on the bench? Does the order of the students on the bench matter here?

10. Ten of your students get in trouble during lunch. Mrs. Worrell tells the students to sit around a circular table. How many different ways can she arrange the students around the table? (Why is this different from #9?) Does the order of the students around the table matter here?
The arrangements below are equivalent for four people sitting at a round table:





11. How many ways can you arrange N distinct items in a line?

12. How many ways can you arrange N distinct items around a circular table?

Icy Rock Dairy has 6 types of ice cream:
Chocolate, Mocha, Pistachio, Raspberry, Strawberry, Vanilla

1. How many different ice cream cones can you make with only one scoop of ice cream?

2. How many different ice cream cones can you make with two different scoops of ice cream, if the order of the scoops **does** matter?

3. How many different ice cream cones can you make with three different scoops of ice cream, if the order of the scoops **does** matter?

4. How many different ice cream cones can you make with four different scoops of ice cream, if the order of the scoops **does** matter?

5. How many different ice cream cones can you make with five different scoops of ice cream, if the order of the scoops **does** matter?

6. How many different ice cream cones can you make with six different scoops of ice cream, if the order of the scoops **does** matter?

7. How many different ice cream cones can you make with two different scoops of ice cream, if the order of the scoops **does not** matter? How does this relate to the answer to #2 (when order does matter)?

8. How many different ice cream cones can you make with three different scoops of ice cream, if the order of the scoops **does not** matter? How does this relate to the answer to #3 (when order does matter)?

9. How many different ice cream cones can you make with four different scoops of ice cream, if the order of the scoops **does not** matter? How does this relate to the answer to #4 (when order does matter)?

10. How many different ice cream cones can you make with five different scoops of ice cream, if the order of the scoops **does not** matter? How does this relate to the answer to #5 (when order does matter)?

11. How many different ice cream cones can you make with six different scoops of ice cream, if the order of the scoops **does not** matter? How does this relate to the answer to #6 (when order does matter)?

12. How many different ice cream cones can you make with k different scoops of ice cream, if the order of the scoops **does** matter, and you have n flavors available?

13. How many different ice cream cones can you make with k different scoops of ice cream, if the order of the scoops **does not** matter, and you have n flavors available?

Combinations and Permutations

Definitions:

- A set of size n is called an **n-set**.

- A **permutation** of an n-set is an arrangement of the elements of the set in order. This is like #1 – 6.

The number of permutations of an n-set is

- Given an n-set, suppose that we want to pick out k elements and arrange them in **order**. Such an arrangement is called a **k-permutation of the n-set**. The number of such permutations is denoted $_nP_k$ or **P(n,k))**. (Your calculator most likely uses $_nP_k$.) This is like #1 – 6.

P(n,k) =

- A **k-combination of an n-set** is a selection of k elements from the set, where **order does NOT matter**. The number of such combinations is sometimes denoted $_nC_k$ or **C(n,k)**. (Your calculator most likely uses $_nC_k$.) However, we will use the notation $\binom{n}{k}$, which reads "**n choose k**." This is like #7 – 11.

$\binom{n}{k}$ =

Calculate the following numbers by hand. Show simplifications. You may verify your answer using your calculator.

1. P(5, 2) =

2. P(7, 3) =

3. P(7, 0) =

4. P(7, 7) =

5. P(n, 0) =

6. P(n ,n) =

7. $\binom{7}{3}$ =

8. $\binom{8}{4}$ =

9. $\binom{9}{9}$ =

10. $\binom{6}{0}$ =

11. $\binom{6}{1}$ =

12. $\binom{n}{0}$ =

13. $\binom{n}{1}$ =

14. $\binom{n}{n}$ =

Pizza Problems

1. Mondo Pizza has 5 different toppings (Anchovies, Bacon, Clams, Eggplant, and Figs). How many different pizzas can you make using these toppings?

2. Suppose the pigs went on strike and Mondo has a bacon shortage, so they only have 4 toppings for pizzas now. How many different pizzas can they make now?

3. What happens if they only had 3 toppings to work with (i.e. say the henplant stopped producing eggplants)?

4. For the pizzas that you found above, where you had five available toppings to work with, how many of those pizzas had zero toppings? How many had exactly one topping? How many had exactly two toppings? 3 toppings? 4 toppings? 5 toppings?

5. Do the same as you did in #4 for when you only have 4 toppings to work with (i.e. when you had a bacon shortage) or when you have 3 toppings to work with (i.e. when all the mushrooms were rotten and you had no bacon).

6. When there are n toppings to work with, how many different types of pizza can be made?

7. When there are n toppings available, how many k topping pizzas can be made?

Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

You are all familiar with Pascal's Triangle seen above. We will go over a few applications of Pascal's Triangle, which you may or may not know. Before we get into that, **list 5 observations** you have about Pascal's Triangle.

1.

2.

3.

4.

5.

Binomial Coefficients:

More Sample Problems for Combinations and Permutations

1. Go back to the "Beginning Sample Problems for Combinations and Permutations" on pages 25 and 26. Answer questions 1 – 5, 7 – 9 in terms of P(n, k) and $\binom{n}{k}$.

In each of the following problems, write your answer in terms of P(*n, k*) or $\binom{n}{k}$ for some *n* and *k*.

(E.g. The number of ways to pick 5 of the available 8 ingredients for a salad is $\binom{8}{5}$ since it does not matter what order I choose these ingredients.)

2. Suppose you have 10 students who are entering a pie-eating contest. How many ways can we choose the winner, runner-up and 2^{nd} runner-up?

3. Suppose the same 10 students are still eating pie. The four best eaters go on to the state competition. How many ways can I choose the 4 students to go on the trip?

4. The school is sending 4 of the 10 students to the state competition, and we know that Davie Hogan and Kobayashi are guaranteed to make it. How many ways can I choose the group to represent the school now?

5. On "Rock of Love", Mr. Michaels has narrowed down the competition to 7 "lovely ladies." He is choosing 3 of them to go riding in the sand dunes with him. How many ways can he choose the three women to go with him?

6. After the trip, Mr. Michaels has decided to go on 4 different trips (Malibu, Monterrey, Carmel, and Tijuana), each with one (different) woman. How many ways can he choose the women to go on these trips?

7. A committee is to be chosen from a set of 7 women and 4 men. How many ways are there to form the committee if:

a. The committee has 5 people, 3 women and 2 men.

b. The committee can be any positive size but must have equal numbers of women and men.

c. The committee has 4 people and one of them must be Mr. Baggins.

d. The committee has 4 people, two of each sex, and Mr. and Mrs. Baggins cannot be on the same committee.

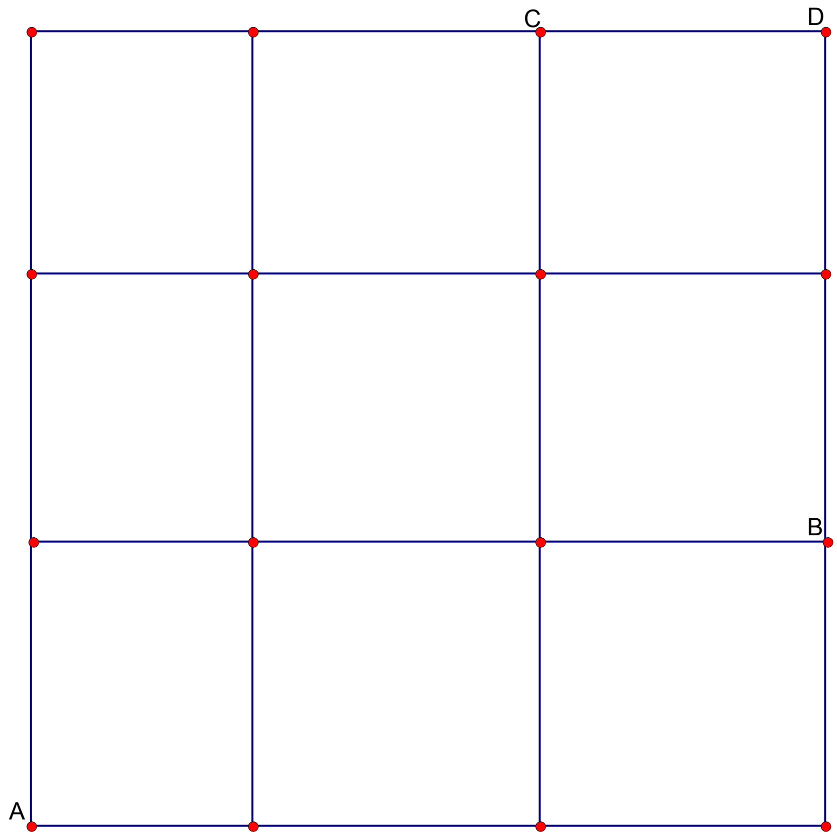
8. There are six different French books, eight different Russian books, and five different Spanish books. How many ways are there to arrange the books in a row on a shelf with all books of the same language grouped together?

9. How many 10-letter "words" are there using 5 different vowels and 5 different consonants (A, E, I, O, and U are vowels)?

10. How many ways are there to pair off 10 women at a dance with 10 out of 20 available men?

Walking the Streets

Consider the following blocks of Portland.



a. How many routes are there directly from A to B? (You CANNOT go above B or retrace your steps.)

b. How many routes are there directly from A to C?

c. How many routes are there directly from A to D?

d. Do you see any relation between these numbers and binomial coefficients (i.e. numbers in Pascal's triangle)? Why does this happen? (You may find it helpful to list the routes for parts and b.)

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Multinomial Coefficients

So far we have mainly been selecting a single subset of objects. We may sometimes wish to do something like the following:

How many ways are there to arrange the letters of the word LEVEL?

(Note: you cannot distinguish between the 1st and last L's in LEVEL. Similarly, you cannot distinguish the E's.)

We call the number of ways to do this a **multinomial coefficient** and denote this quantity as $\binom{5}{2,2,1}$ (for this particular example).

Where do you think each of those numbers came from?

Let's derive a general formula for the multinomial coefficient.

Suppose you have n_1 objects of one type, n_2 objects of a 2nd type, …., n_k objects of a last type, where $n_1 + n_2 + \cdots + n_k = n$. You want to put these n objects in a line. How many ways can you do this? (This is like having 2 L's, 2 E's and 1 V that you want to put in a line.)

Answer: The number of ways to do this is;

$\binom{n}{n_1, n_2, \ldots, n_k}$ =

Do the following exercises:

1. The university's registrar's office is having a problem. It has 11 new students to squeeze into 4 sections of an introductory course: 3 in the first, 4 each in the second and third, and 0 in the fourth (that section is already full). In how many ways can this be done?

2. A code is being written using the five symbols +, #, √, ∇, ×.
   a. How many 10-digit codewords are there that use exactly 2 of each symbol?

   b. If a 10-digit codeword is chosen at random, what is the probability that it will use exactly 2 of each symbol?

3. How many ways are there to arrange the letters of the word *Mississippi*?

Lottery Games

1. In the daily Play 4, players select a number between 0000 and 9999. The lottery machine contains 4 bins, each with 10 ping-pong balls. Each bin has its ping-pong balls labeled 0, 1, 2, …, 9. The state then “randomly” selects a ball from each bin and forms a number between 0000 and 9999.

   a. What is the probability you select the exact “4-digit” number?

   b. You can choose to play a “BOX”, where if your numbers match in **any** order, then you win.

      i. If you select to play “BOX” and your number is 1112, what is the probability that you win?

      ii. If you select to play “BOX” and your number is 1122, what is the probability that you win?

      iii. If you select to play “BOX” and your number is 1123, what is the probability that you win?

      iv. If you select to play “BOX” and your number is 1234, what is the probability that you win?

2. Looking back in #1 (part b),

   a. How many winning numbers have the form 1112?

   b. How many winning numbers have the form 1122?

   c. How many winning numbers have the form 1123?

   d. How many winning numbers have the form 1234?

3. In Powerball, you select 5 (different) numbers between 1 and 59. You then select the red Powerball, which is between 1 and 39. The Powerball Gods then select 5 different numbers between 1 and 59 and the Powerball. You win the Jackpot if you match all the Powerball Gods' numbers, including the red Powerball. (The order of the non-red numbers DOES NOT MATTER!)

   a. What is the probability that you match the (red) Powerball?

   b. What is the probability that you win the Jackpot?

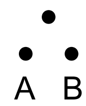
   c. What is the probability that you match exactly 4 numbers and the Powerball?

   d. If the Jackpot is over $250,000,000, then how come Bill Gates doesn't just buy all possible tickets and win the Jackpot? (Ignore the tax issue and the morality issue.)

Plinko!

Consider the following severely modified Plinko! games. We want to know the probability of landing on each spot. We will do Game 4 together first.

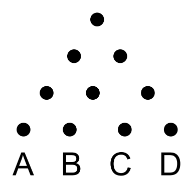
Game 1



{Let Pr(A) = Prob(Landing on A)}

Pr(A) = ____________ Pr(B) = ________________

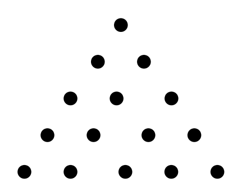
Game 2



Pr(A) = ___________ Pr(B) = ____________

Pr(C) = ____________ Pr(D) = ____________

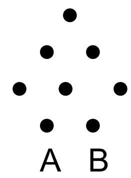
Game 3



Pr(A) = ___________ Pr(B) = ____________

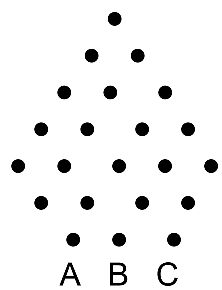
Pr(C) = ____________ Pr(D) = ____________ Pr(E) = ______________

Game 4



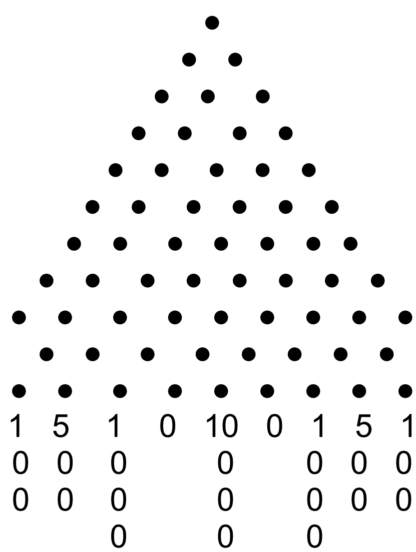
Pr(A) = ____________ Pr(B) = _____________

Game 5



Pr(A) = ____________ Pr(B) = _____________ Pr(C) = _____________

Real Plinko!



ways to get left $100 = ___________

ways to get $10,000 = _______________

The Sierpinksi Triangle

A **fractal** is a rough or fragmented shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole. (Wikipedia)

Follow these steps to create a fractal called "The Sierpinski Triangle".

Before you start making the fractal, draw an 8″ equilateral triangle on a standard sheet of paper,

1. Connect the midpoints of the triangle. Shade in this new triangle whose vertices are the midpoints of the original triangle. (You should have four 4″ equilateral triangles, where one of the triangles is shaded.)

   Look at the picture you made in Step 1. What fraction of the original 8″ equilateral triangle did you NOT shade? ________________________

2. For each of the triangles, which are not shaded, connect the midpoints of the triangle and shade in this new triangle. (These new triangles should be 2″ equilateral triangles.)

   Look at the picture you made in Step 2. What fraction of the original 8″ equilateral triangle did you NOT shade? _________________________

3. For each of the triangles, which are not shaded after Step 3, connect the midpoints of the triangle and shade in this new triangle. (These new triangles should be 1″ equilateral triangles.)

   Look at the picture you made in Step 3. What fraction of the original 8″ equilateral triangle did you NOT shade? _________________________

4. Do you see a pattern here? Use the pattern to predict the fraction of the triangle you would NOT shade in the Step 4 Triangle. Confirm your prediction and explain.

5. Develop a formula so that you could calculate the fraction of the area, which is NOT shaded for any step.

Triangle Game

Materials: A ruler, a 6-sided die, overhead pen, and the Triangle game overhead

Directions:

1. Randomly select a point inside/on your triangle provided. Put a dot there.

2. Roll the provided die. If you roll a 1 or 2, put a dot EXACTLY halfway to the Red Point. If you roll a 3 or 4, put a dot EXACTLY halfway to the Green Point. If you roll a 5 or 6, put a dot EXACTLY halfway to the Blue Dot. (We will call this, Dot 1.) **Be sure that you actually measure. No estimating allowed!**

3. Erase your first dot ONLY. Keep all other dots you draw from now on.

4. Roll the die again. Put a dot EXACTLY halfway between Dot 1 and the appropriate vertex of the triangle. We will call this Dot 2. **Be sure that you actually measure. No estimating allowed!**

5. Repeat Step 4, but measure from Dot 2 instead of Dot 1.

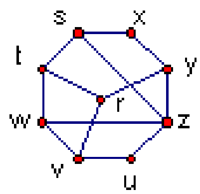
6. Continue this process until you have 30 dots on your paper.

7. When you are done, bring your overhead to the front.

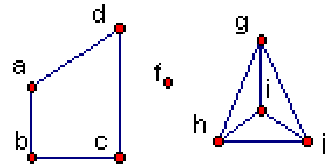
We will now visit the following website to see a computer simulation of our game.

http://www.shodor.org/interactivate/activities/TheChaosGame/

Introduction to Graph Theory



G=(V,E)



H=(V,E)

We'll start with a series of definitions.

- A **graph** G consists of a (usually) finite set V of **vertices** (think points) and a set E of **edges** that connect two (not necessarily distinct) vertices in V. We write G=(V,E) to denote the graph with vertex set V and edge set E.

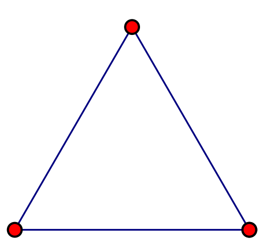
- The number of vertices in a graph G=(V,E) is called the **order** of the graph. Note the order of G=(V,E) is |V|.

- For a graph G=(V,E), we say two vertices $u, v \in V$ **adjacent** if uv is an edge in E.
  e.g.

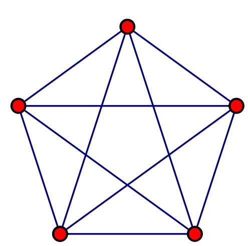
- The **degree** of a vertex *v* is the number of edges incident to it and is denoted deg(*v*).
  e.g.

- A graph G is **connected** provided each pair of vertices is connected by a sequence of edges. Otherwise, G is **disconnected**.

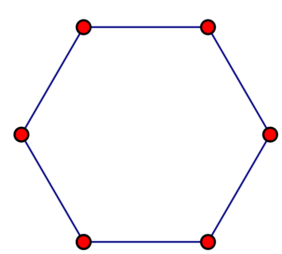
- A **complete graph** on n vertices, denoted K_n is a graph where each pair of vertices is connected by exactly one edge.



K_3

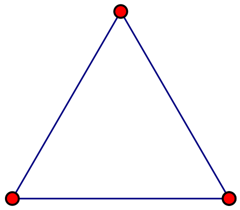


K_5

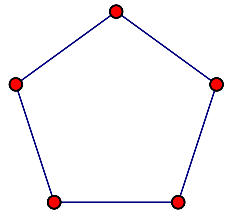


K_6

- A **cycle** on n vertices (v_1, v_2, …, v_n), denoted C_n is a graph where v_1 is connected to v_2, v_2 is connected to v_3, …, v_{n-1} is connected to v_n, and v_n is connected back to v_1.



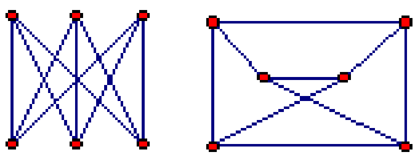
C_3

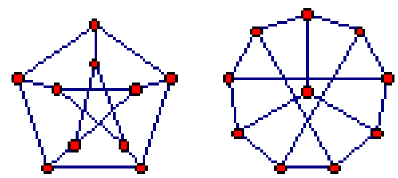


C_5

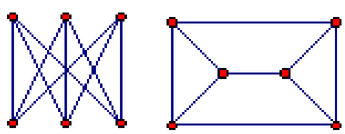
- Two graphs $G=(V_1,E_1)$ and $H=(V_2,E_2)$ are called **isomorphic** if they are "exactly the same, but different." (We write $G \cong H$.) This just means that if you had a drawing program and drew G, you could re-label the vertices (to match H's vertex set) and drag the vertices around to end up at H.

Examples: Each pair of graphs below are isomorphic.





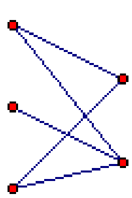
- The graphs below are NOT isomorphic. (Why?)

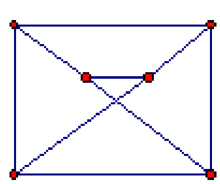


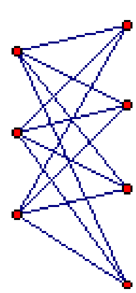
- A **bipartite graph** is a graph $G=(V,E)=(X \cup Y, E)$ in which the vertex set V can be partitioned into two sets X and Y such that each edge has one vertex in X and one vertex in Y. (i.e. there are no edges between two vertices in X and no edges between two vertices in Y).

  A bipartite graph $G = (X \cup Y, E)$ is called **complete** if each vertex in X is adjacent to each vertex in Y (and vice versa). We write $K_{m,n}$ to denote the bipartite graph $G=(X \cup Y, E)$ when $|X|=m$ and $|Y|=n$.

Examples







QUESTIONS

1. How many edges are there in K_n?

2. In a simple graph G on *n* vertices, what are the possible values for d(*v*), where *v* is a vertex of G?

3. Prove the following statement: Every simple graph on at least two vertices has two vertices of equal degree.

4. If G has a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, what is $d(v_1) + d(v_2) + \ldots + d(v_n)$ compared to the number of edges?

5. Can a graph have an odd number of odd degree vertices? Why or why not?

6. How many edges are there in C_n?

7. How many non-isomorphic graphs can be formed on three vertices? On four vertices?

8. Can a bipartite graph have an odd cycle? Why or why not?

9. A graph is **k-regular** if each vertex has degree k. Show that the number of vertices in a *k*-regular graph is even if *k* is odd.

Graph Coloring

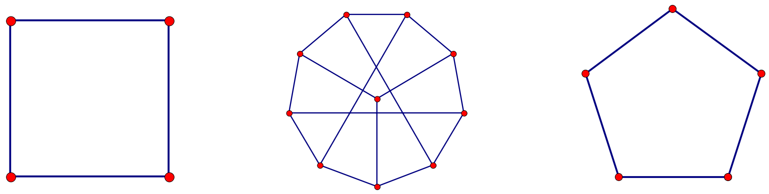
Vertex Coloring:

A **proper k-coloring** of a graph G is a labeling $f: V(G) \to \{1, ..., k\}$ such that if xy is an edge in G, then $f(x) \neq f(y)$ (meaning if two vertices are adjacent, then they are colored differently).

A graph G is **k-colorable** if it has a proper **k-coloring**.

The **chromatic number** $\chi(G)$ ("chi of G") is the minimum *k* such that G is *k*-colorable.

1. Find the chromatic numbers of the following graphs.

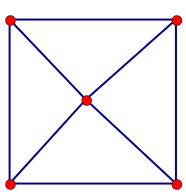


2. Find $\chi(K_n)$ (where K_n is a graph on n vertices where each pair of vertices is connected by an edge).

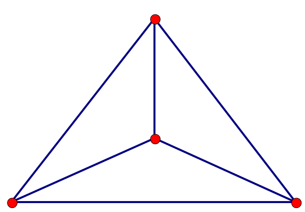
3. Find $\chi(C_n)$ (where C_n is the cycle on n vertices).

Let W_n be a graph consisting of a cycle of length n and another vertex *v* (called the hub), where the hub is connected to every other vertex.

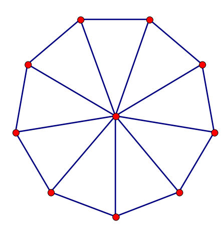
Examples of Wheels:



W_4



W_3

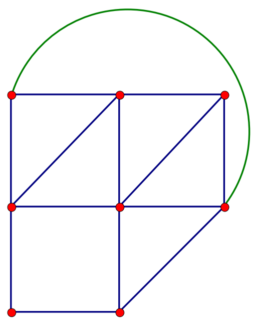


W_9

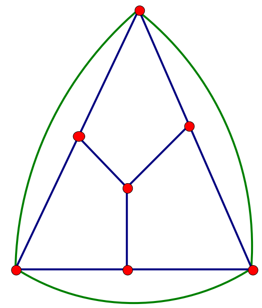
4. Find $\chi(W_n)$.

5. Find the chromatic number for each of the following graphs. Try to give an argument to show that fewer colors will not suffice.

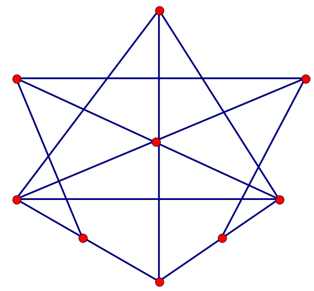
a.



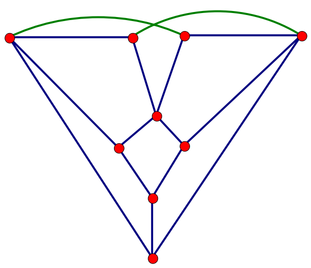
b.



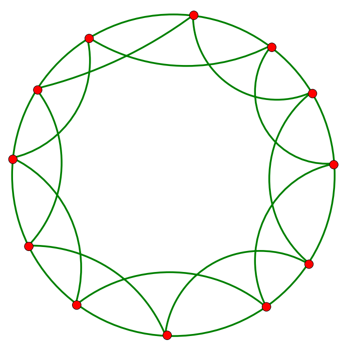
c.



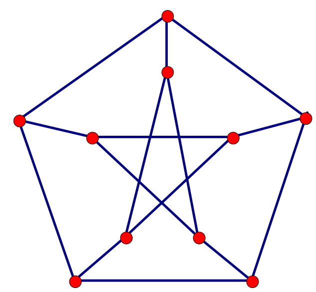
d.



e.



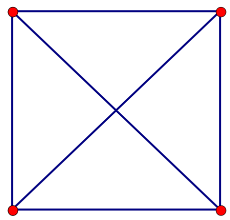
f.



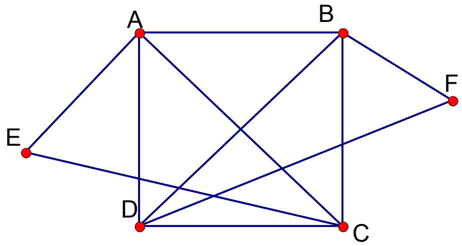
Planar Graph Theory

We say that a graph is **planar** if it can be drawn in the plane without edges crossing. We use the term **plane graph** to refer to a planar depiction of a planar graph.

Example 1: K_4 is a planar graph. Find a plane graph version of K_4.



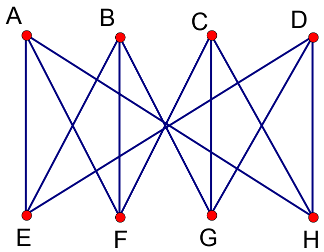
Example 2: The following is also planar. Find a plane graph version of the graph.



A Method that sometimes works for drawing the plane graph for a planar graph:

1. Find the largest cycle in the graph.
2. The remaining edges must be drawn inside/outside the cycle so that they do not cross each other.

Example 3: Using the method above, find a plane graph version of the graph below.

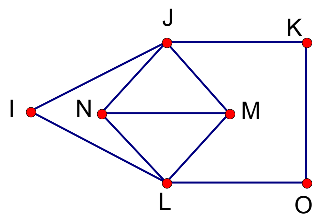


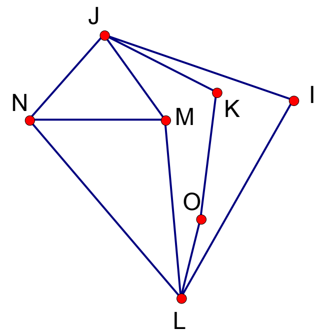
non e.g.

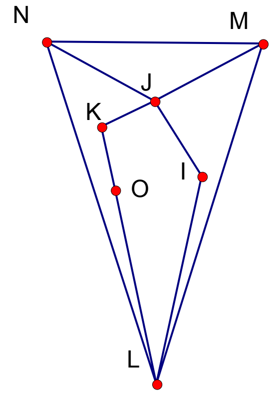
$K_{3,3}$: K_5

A Magic Formula

Here are three (plane graph) depictions of the same planar graph:







A **face** of a plane graph is a region enclosed by the edges of the graph. There is also an unbounded face, which is the outside of the graph.

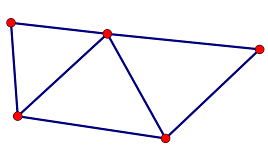
1. For each of these graphs,

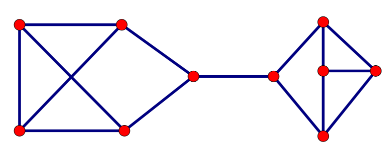
   V = # of vertices of the graph = ______________

   E = # of edges of the graph = _______________

   F = # of faces of the graph = ________________

2. For each of the graphs below, determine V, E, and F.





V = ____ E = ____ F = _____ V = _____ E = _____ F = ______

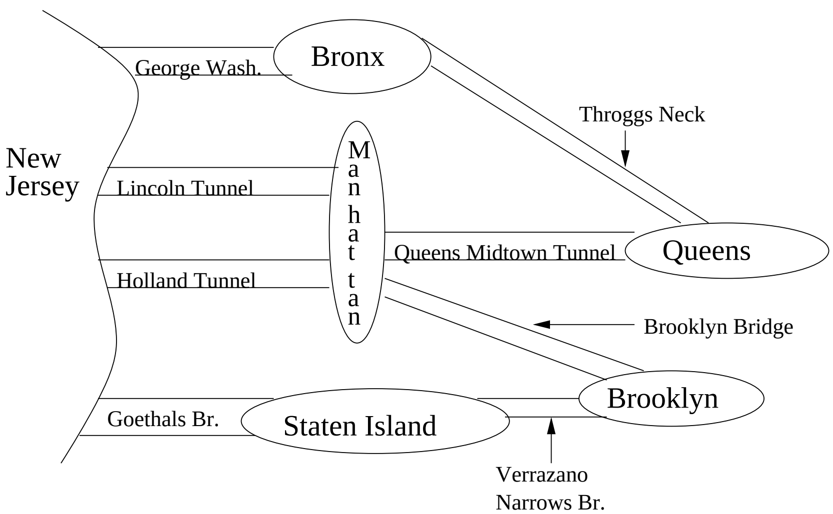
3. Do you have a conjecture for an equation relating V, E and F for any plane graph G?

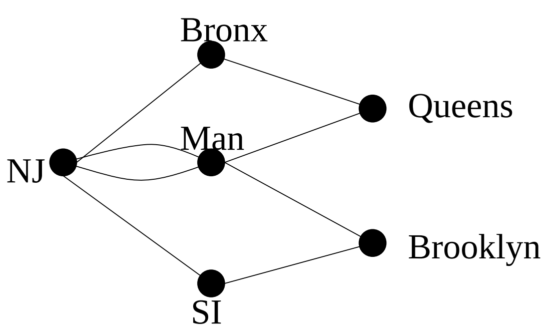
   This is called ____________________

4. For each of the 5 Platonic Solids, find the # of vertices, # of edges and # of faces. Now find an equation relating V, E and F for the Platonic Solids.

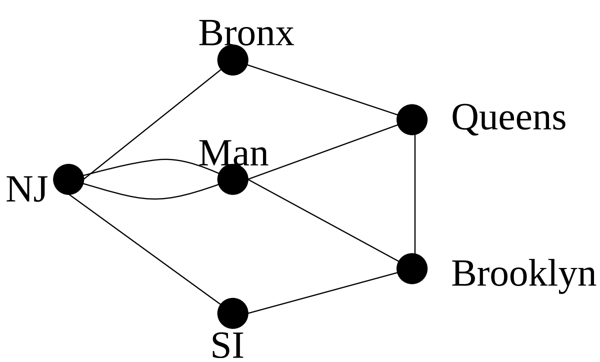
Pizza Delivery Problem

Below is a map of New York City and its surrounding area. Suppose you are a Domino's delivery person in New Jersey. Your boss tells you to deliver a pizza to a toll collector for **each** of the bridges in the map. In order to save time, you want to do this so that <u>you cross each bridge exactly once</u> and return back to New Jersey. Can you do this? If so, show the route you would take. The graph below models the map given.





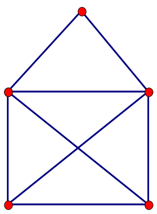
What happens if we add the I-495 (from Brooklyn to Queens) as shown below? Can you start in New Jersey and cross each bridge exactly once? Can you start anywhere else and cross each bridge exactly once?



When you start from a vertex in a graph, walk along every edge exactly once, and return to the starting vertex, we say this graph has an **Euler Circuit**. If you walk along every edge, but do not return to the starting vertex, we say the graph has an **Euler Trail**.

For the following graphs, determine if they have (a) an Euler circuit, (b) an Euler trail (but not an Euler circuit) or (c) neither.

Goal: By looking at a graph, one can tell if it has an Euler circuit/trail or not. What makes a graph have an Euler circuit/trail? **Hint**: Look at the degrees of the vertices.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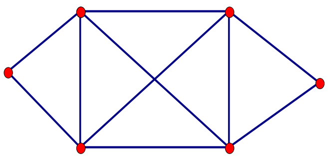
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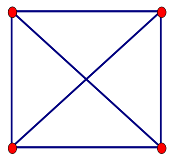
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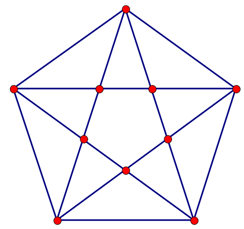
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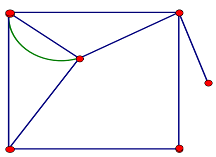
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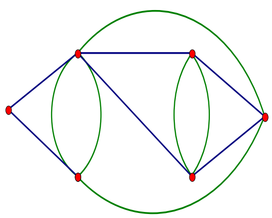
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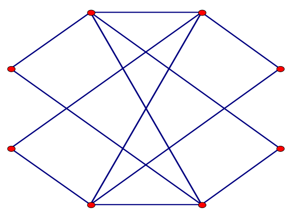
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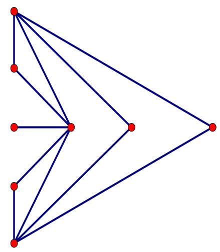
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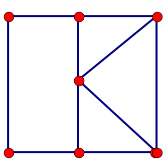
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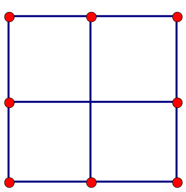
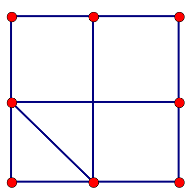
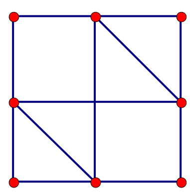
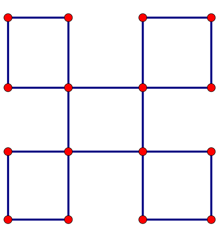

____X.


______XI.


XII.________________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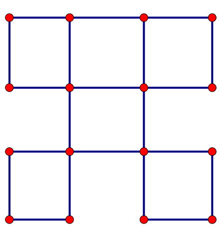
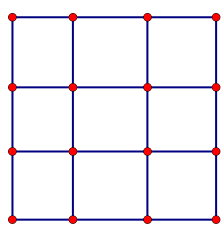
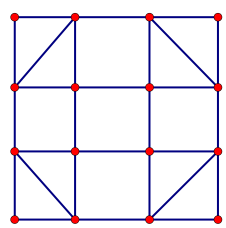

_XIV.


XV.


XVI.____

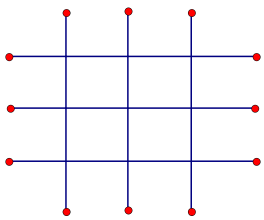
A graph has an Euler circuit when ______________________________________.

A graph has an Euler trail when ____.

Winning Lines in Tic-Tac-Toe

1. The standard Tic-Tac-Toe is played on a 3 x 3 board, where there are ____ vertical winning lines, _______ horizontal winning lines, ______ diagonal winning lines. This is a grand total of _________ winning lines in the standard 3x3 Tic-Tac-Toe.

2. In 4 x 4 Tic-Tac-Toe, the winning lines are again 4-in-a-row horizontally, vertically or diagonally. How many winning lines are there in 4 x 4 Tic-Tac-Toe? _____________________



3. How many winning lines are there in 5x5 Tic-Tac-Toe? _____________

4. How many winning lines are there in *n* x *n* Tic-Tac-Toe? ____________

You may know about three-dimensional Tic-Tac-Toe, where the winning lines either have one, two or three coordinates varying. In the interest of my sanity, we will say *n* x *n* x *n* Tic-Tac-Toe is $\mathbf{n^3}$ **TTT**. This game will be played on a cube where each side is of length *n*.

5. How many winning lines are there in 2^3 TTT?

6. How many winning lines are there in 3^3 TTT?

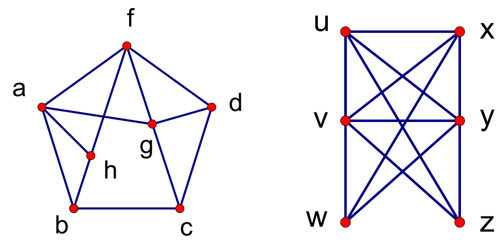
7. How many winning lines are there in 4^3 TTT?

8. Find a formula for the number of winning lines in n^3 TTT.

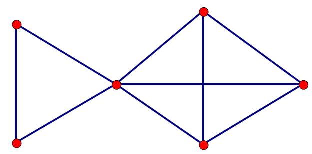
Hamilton Graphs and Traveling Salesman Problem

We say a graph has a **Hamilton cycle** if there is a cycle, which goes through every vertex (not necessarily every edge).

Examples:



Non-Example:

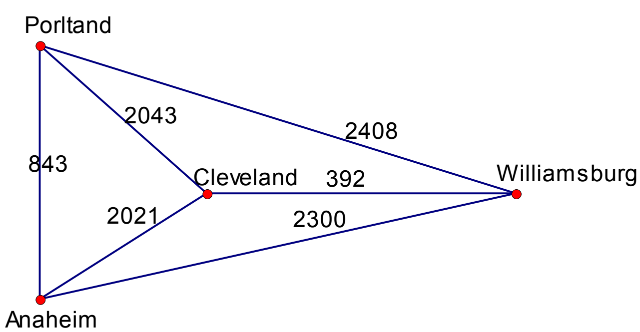


1. How many Hamiltonian cycles does K_n have?

2. **Traveling Salesman Problem**
Amusement Parks: Suppose you live in Portland, OR and want to go on an amusement park tour. You want to visit
Cedar Point in Cleveland, OH
Busch Gardens, Williamsburg, VA
Disney Land in Anaheim, CA
You want to leave from Portland, visit all three parks and then return to Portland. Since you are young and can't afford to fly, you will have to drive the entire time. Below, you will find a table with the mileage between cities. **Try to find the shortest route**. Come up with 2 algorithms that seem like they should work. (An algorithm is a procedure one can follow to find the cheapest route.)

	Cleveland	**Anaheim**	**Portland**	**Williamsburg**
Cleveland, OH		2021	2043	393
Anaheim, CA	2021		843	2300
Portland, OR	2043	843		2408
Williamsburg, VA	392	2300	2408	

Here is a graph representing the data:



Describe Method 1:

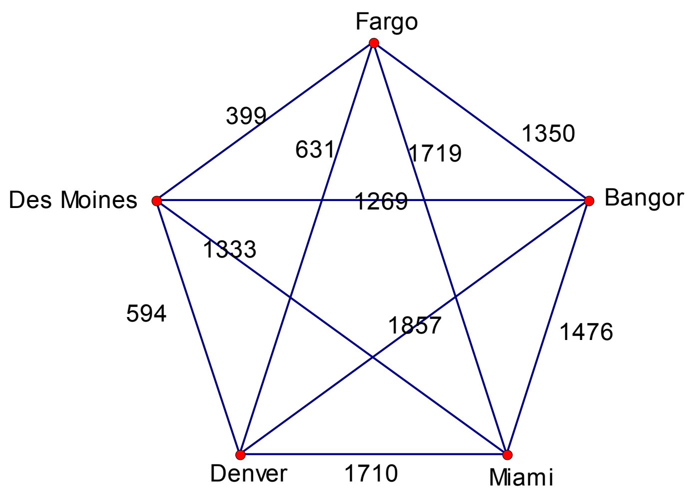
Method 1 Total Mileage: ____________

Describe Method 2:

Method 2 Total Mileage: ____________

Visiting Relatives: Suppose you want to go to visit four of your relatives this summer. You are currently living in Fargo, ND. Unfortunately, you cannot afford to fly everywhere, but your car is in good shape. Therefore you will drive to see each of your relatives. Below is a table listing the distances (in miles) between pairs of cities. Try to find the route that requires you to put the least amount of mileage on your car. You must start in Fargo and end up in Fargo.

	Bangor	Denver	Des Moines	Fargo	Miami
Bangor		1857	1269	1350	1476
Denver	1857		594	631	1710
Des Moines	1269	594		399	1333
Fargo	1350	631	399		1719
Miami	1476	1710	1333	1719	



Describe Method 1:

Method 1 Total Mileage: ____________

Describe Method 2:

Method 2 Total Mileage: ____________

MATH HOMEWORK POLICIES
Kruczek, Mathematics Department, WOU

Overall

- Your homework is a reflection of you as a person, a scholar and a future teacher.
- Your presentation and write up of your homework is an important way for you to communicate your understanding of course topics. This is a math focus course; it is expected that you will turn in only quality work.
- There is no policy for rewriting or resubmitting homework. Do your best work the first time!
- If you need help, ASK. There is no excuse for not "getting" homework and not asking for help. You may seek help during office hours and by email.
- Readability
  - o Please write legibly. If you cannot do this, then type your homework.
  - o Please do not turn in spiral bound paper with little "fringies" on it.

Explanations

- In general, you must explain all of your work. This class is for future teachers and one overall goal is that you will be able to explain how the mathematical topics covered in this class work. Even if a problem does not explicitly say "explain" it is assumed you will explain your solution path.
- If any adult reader reads your work and wonders "why?" about any detail, you need more explanation.
- **Show your work:** Some questions will require steps. In these cases, show all of your steps and work. Don't exclude details you work out on scratch paper.

Assessment

- Each problem will be graded as follows: 25% for the right answer, 25% for the right steps, and 50% for the explanation (clearly showing ALL OF THE STEPS you took to solve the problem).

Working Together on Homework

- I encourage you to work with other people on the homework. I do have some advice and requirements though.
  - o I encourage you to **try to do all of the homework by yourself** first, before working with others. There is no use listening to someone else explain how to solve the problem if you do not have an idea of how to approach it.
  - o After you are done working with others, try to write up the solutions to the questions by yourself. This way, you will be sure to explain in a way that you will understand later.
  - o NO COPYING OTHER STUDENT'S HOMEWORK! Please write up solutions on your own, without the assistance of someone else either telling you what to write or reading word for word what she/he did.