PERMUTATIONS AND COMBINATIONS

Mathematics for Elementary Teachers: A Conceptual Approach

New Material for the Eighth Edition

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Math 212 Extra Credit Assignment
Permutations and Combinations

Up to 15 homework points

All pages are part of the handout “Permutations and Combinations,” Bennett, Burton and Nelson

1. Read the page 540 revision

2. Read the new section materials, “Permutations and Combinations”

3. Do the corresponding exercises based on your group, A or B (assigned in class)

Group	**Exercise #s**	**Chapter Test**
A	48, 50, 52, 54, 56, 58a, 59	16
B	49, 51, 53, 55, 57, 58b, 60	17

4. Turn the solutions to the exercises in by March 12

5. If you have any comments such as “this part is confusing” or “this isn’t clear” etc., please mark the comments directly on the handout and turn the marked handout in with your exercise solutions on March 12.

National Oceanic and Atmospheric Administration's weather satellites sent back the preceding photograph of North and South America showing storms approaching the East Coast of the United States and the Caribbean Islands. Weather forecasts are usually stated in terms of probability, and each probability may be determined from several others. For example, there may be one probability for a cold front and another probability for a change in wind direction. In this section, we will see how to use the probabilities of two or more events to determine the probability of a combination of events.

NCTM Standards

Students should also explore probability through experiments that have only a few outcomes, such as using game spinners with certain portions shaded and considering how likely it is that the spinner will land on a particular color. p. 181

PROBABILITIES OF OUTCOMES

In Section 8.1 we studied **single-stage experiments** such as spinning a spinner, rolling a die, and tossing a coin. These experiments are over after one step. Now we will study combinations of experiments, called **multistage experiments**.

Suppose we spin spinner A and then spinner B in Figure 8.7. This is an example of a *two-stage experiment*.

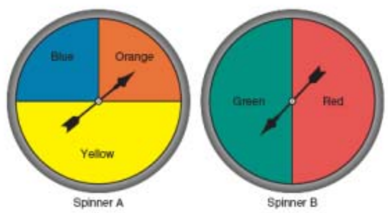


Figure 8.7

The different outcomes for multistage experiments can be determined by constructing *tree diagrams*, which were used in Chapter 3 as a model for the multiplication of whole numbers. Since there are 3 different outcomes from spinner A and 2 different outcomes from spinner B, the experiment of spinning first spinner A and then spinner B has $3 \times 2 = 6$ outcomes as shown in figure 8.8. This figure illustrates the following generalization.

> **Multiplication Principle** If event A can occur in m ways and then event B can occur in n ways, no matter what happens in event A, then event A followed by event B can occur in $m \times n$ ways.

The Multiplication Principle can be generalized to products with more than two factors. For example, if a third spinner C with 5 outcomes is added to figure 8.7, then the total number of outcomes for spinning spinner A followed by spinner B followed by spinner C is $3 \times 2 \times 5 = 30$ outcomes.

Figure 8.9 shows the probabilities of obtaining each color and each outcome in Figure 8.8. Such a diagram is called a probability tree. The probability of each of the 6 outcomes can be determined from this probability tree. For example, consider the probability of obtaining BR (blue on spinner A followed by red on spinner B). Since blue occurs $\frac{1}{4}$ of the time and red occurs $\frac{1}{2}$ of the times that blue occurs, the probability of BR is $\frac{1}{4} \times \frac{1}{2}$ or $\frac{1}{8}$. This probability is the product of the two probabilities along the path that leads to BR. Similarly, the probability of YG (yellow followed by green) is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Notice that the sum of the probabilities for all 6 outcomes is 1.

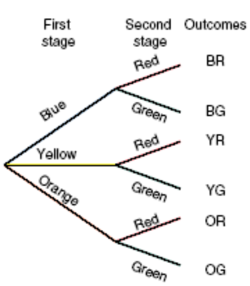


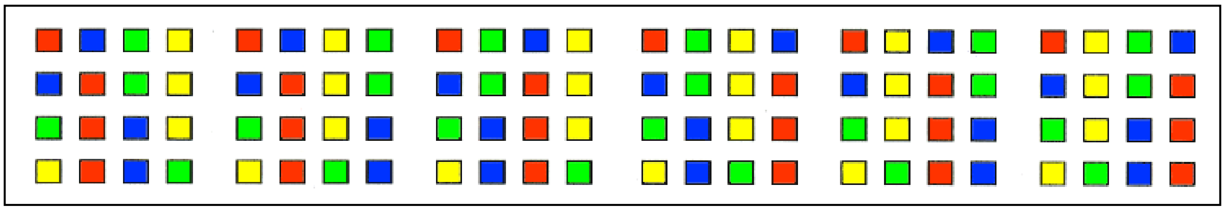
Figure 8.8

Permutations and Combinations

In Section 8.1 and the first part of Section 8.2, we were able to determine probabilities by listing the elements of sample spaces and by using tree diagrams. Some sample spaces have too many outcomes to conveniently list so we will now consider methods of finding the numbers of elements for larger sample spaces.

EXAMPLE I How many different ways are there to place four different colored tiles in a row? Assume the tiles are red, blue, green and yellow.

Solution One method of solution is to place the four colored tiles in all possible different orders. There are 24 different arrangements as shown here.



The *Multiplication Principle* (see page 540) can be used for a more convenient solution. Sketch four blank spaces and imagine placing one of the four tiles in each of the spaces:

______	______	______	______
1st tile	*2nd tile*	*3rd tile*	*4th tile*

Any one of the four tiles can be placed in the first space; any one of the three remaining tiles can be placed in the second space; any one of the two remaining tiles can be placed in the third space; and the remaining tile can be placed in the fourth space. By the *Multiplication Principle*, the number of arrangements is $4 \times 3 \times 2 \times 1 = 24$

□

Example I illustrates a permutation. A **permutation** of objects is an arrangement of these objects into a *particular order*.

Notice that the solution for Example I involves the product of decreasing whole numbers. In general, for any whole number $n > 0$, the product of the whole numbers from 1 through n is written as n! and called **n factorial**. It is usually more convenient to write n! with the whole numbers in decreasing order. For example, in Example I, $4! = 4 \times 3 \times 2 \times 1 = 24$.

Technology Connection

Many calculators have a factorial ! key. To use this key, enter n, enter ! and enter =

n factorial $n! = n \times n - 1 \times \ldots \times 2 \times 1$

Special Case: 0! is defined to be 1.

EXAMPLE J How many different ways are there to place three different colored tiles chosen form a set of five different colored tiles in a row? Assume the five tiles are red, blue, green, yellow and orange.

Solution Using the *Multiplication Principle* and three blank spaces;

$$\underset{\textit{1st tile}}{\underline{\qquad\qquad}} \quad \underset{\textit{2nd tile}}{\underline{\qquad\qquad}} \quad \underset{\textit{3rd tile}}{\underline{\qquad\qquad}}$$

any one of five tiles can be placed in the first space, any one of the 4 remaining tiles in the second space and any one of the three remaining tiles in the third space. So there are 5 × 4 × 3 = 60 different arrangements or permutations of 5 colored tiles taken 3 at a time. The number of permutations of 3 objects from a set of 5 objects is abbreviated as ${}_5P_3$ and we have shown here that ${}_5P_3 = 60$

□

Notice that the solution to Example J can also be expressed as follows by using factorials:

$${}_5P_3 = 60 = 5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2 \times 1}{2 \times 1} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

This expression is a special case of the following formula for determining the number of permutations of ***n*** objects taken ***r*** at a time:

> Technology Connection
>
> Many calculators have a permutations $_nP_r$ key. To use this key, enter n, enter $_nP_r$, enter r and enter =

> **Theorem** The number of permutations of ***n*** objects taken ***r*** objects at a time, where $0 \le r \le n$, is
>
> $${}_nP_r = \frac{n!}{(n-r)!}$$

The word "permutations" in the question in Example J will usually not be given, as illustrated in the next example.

EXAMPLE K In a school soccer league with seven teams, in how many ways can they finish in the positions "winner", "runner-up" and "third place?"

Solution In forming all the possible arrangements for the three finishing places, order must be considered. For example, having Team 4, Team 7 and Team 2 in the positions of winner, runner-up and third place, respectively, is different from having Team 7, Team 2, and Team 4 in these three finishing spots. Using the *Multiplication Principle*, there are 7 possibilities for the winner, and then 6 possibilities are left for the runner-up, and then 5 possibilities are left for the third place. So, there are 7 × 6 × 5 different ways the teams can finish in the positions of winner, runner-up and third place.

$$\underset{\textit{Winner}}{\boxed{7}} \times \underset{\textit{Runner Up}}{\boxed{6}} \times \underset{\textit{Third Place}}{\boxed{5}} = \quad 210 \text{ possibilities}$$

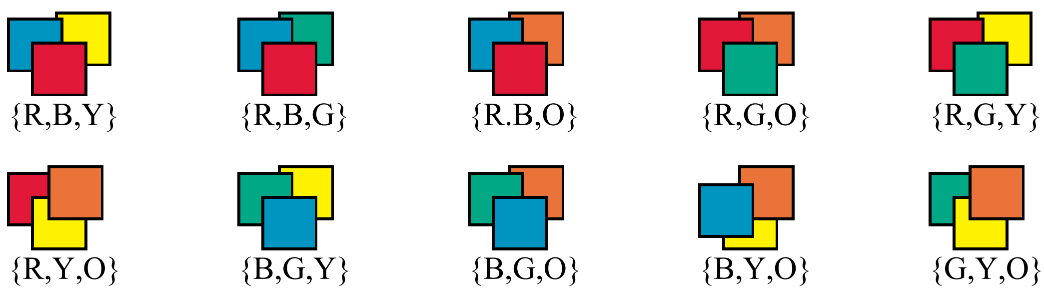
Using the permutation formula for 7 teams taken 3 at a time,

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1} = 7\times 6\times 5 = 210$$

In *permutations* the order of the elements is important. However, in forming collections order is sometimes not important and can be ignored. A collection of objects for which order is not important is called a **combination**. Consider the following example using the five different colored tiles from Example J.

EXAMPLE L How many different collections of three tiles can be chosen from a set of five different colored tiles (red, blue, green, yellow, and orange).?

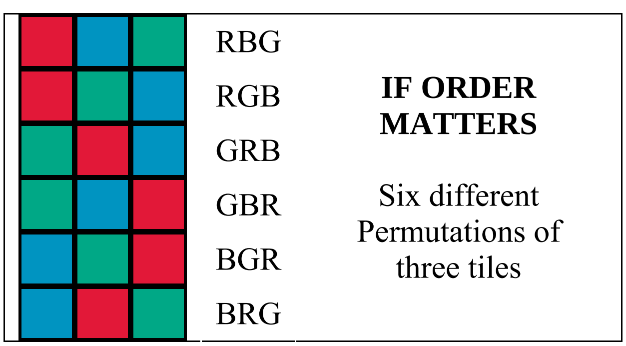
Solution Since order is not important here we can systematically list the different combinations to see there are 10 distinct combinations.

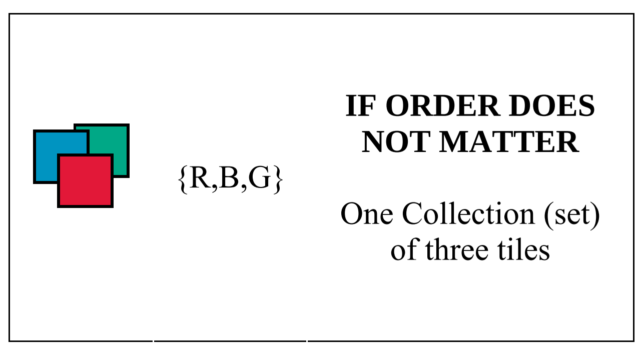


Note that each collection of three tiles is given as a set to indicate the order of the tiles does not matter.

The number of combinations of 3 things chosen from a collection of 5 objects can be abbreviated as $_5C_3$ and we have shown here that $_5C_3 = 10$

It is interesting to notice that the three tiles from each of the above combinations can be arranged in 3! = 3 × 2 × 1= 6 ways. For example, the three tiles from the first combination {R,B,G} can be arranged, if order matters, in six different permutations:





Since each collection of three tiles can be arranged 3! ways this leads to the following observation that

$$3! \times {}_5C_3 = {}_5P_3 \quad \text{or} \quad {}_5C_3 = \frac{{}_5P_3}{3!} = \frac{5!}{(5-3)!3!}$$

□

This expression is a special case of the following formula for determining the number of *combinations* of ***n*** things taken ***r*** at a time.

> **Technology Connection**
> Many calculators have a combinations $_nC_r$ key. To use this key, enter n, enter $_nC_r$, enter r and enter =

> **Theorem** The number of combinations of ***n*** objects taken ***r*** objects at a time, where $0 \le r \le n$, is
>
> $${}_nC_r = \frac{n!}{(n-r)!r!}$$

Examples J and L show that the number of permutations of 5 objects taken 3 at a time is 6 times the number of combinations of 5 objects taken 3 at a time. In general, for ***n*** objects taken ***r*** at a time, there will be more permutations than combinations because considering the different orders of objects increases the number of outcomes.

The first step in solving problems involving permutations or combinations is determining whether or not it is necessary to consider the order of the elements. The following two questions in Example M will help to distinguish between when to use permutations and when to use combinations.

EXAMPLE M The school hiking club has 10 members.
(1) In how many ways can 3 members of the club be chosen for the Rules Committee?
(2) In how many ways can 3 members of the club be chosen for the offices of President, Vice President and Secretary?

Solution
(1) There is no requirement to consider the order of the people on the 3-person Rules Committee. So the number of different committees can be found with the formula for combinations.

$${}_{10}C_3 = \frac{10!}{(10-3)!3!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(7\times6\times5\times4\times3\times2\times1)\times(3\times2\times1)} = \frac{10\times9\times8}{6} = 120$$

(2) Order must be considered in choosing the three officers because it makes a difference as to whom holds each office. So the number of different possibilities for the three offices can be found with the formula for permutations.

$${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = 10\times9\times8 = 720$$

□

The following two examples use permutations and combinations to find probabilities. These two examples are very similar, but you may be surprised at their different probabilities.

EXAMPLE N If 5 students are randomly chosen from a group of 12 students for the offices of President, Vice President, Secretary, Treasurer and Activity Director, what is the probability that group members Alice and Tom will be chosen for Secretary and Activity Director, respectively?

Solution For any 5 students that are selected, it matters which students hold the various offices. That is, order is important so the problem involves permutations.

1) The total number of different permutations of 12 students taken 5 at a time is:

$$_{12}P_5 = \frac{12!}{(12-5)!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1} = 12 \times 11 \times 10 \times 9 \times 8$$

2) If Alice is to be Secretary and Tom is to be Activity Director, the 3 remaining students can be chosen in $_{10}P_3$ different ways.

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1} = 10 \times 9 \times 8$$

Since there is only way to choose Alice to be Secretary and Tom to be Activity Director there are

$1 \times {}_{10}P_3 = 1 \times 10 \times 9 \times 8$
ways to pick a Rules Committee with Alice as Secretary and Tom as Activity Director

3) Using the results from 1) and 2) the probability that Alice will be Secretary and Tom will be Activity Director is:

$$\frac{10\times 9\times 8}{12\times 11\times 10\times 9\times 8} = \frac{1}{12\times 11} = \frac{1}{132} \approx .0075 \approx 1\% \text{ to the nearest percent}$$

EXAMPLE O If 5 students are to be randomly chosen from a group of 12 students to form a committee for a class trip, what is the probability that group members Alice and Tom will be chosen for the committee?

Solution For any 5 students that are selected, the order of the students does not matter. That is, order is not important so the problem involves combinations.

1) The total number of different combinations of 12 students taken 5 at a time is:

$$_{12}C_5 = \frac{12!}{(12-5)!5!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(7\times 6\times 5\times 4\times 3\times 2\times 1)\times(5\times 4\times 3\times 2\times 1)} = \frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2\times 1} = 792$$

2) If Alice and Tom are to be on the committee, the 3 remaining students can be chosen in ${}_{10}C_3$ different ways.

$$ {}_{10}C_3 = \frac{10!}{(10-3)!3!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(7\times6\times5\times4\times3\times2\times1)\times(3\times2\times1)} = \frac{10\times9\times8}{3\times2\times1} = 120 $$

3) Using the results from 1) and 2) the probability that Alice and Tom will be on the committee is:

$$ \frac{120}{792} \approx .15 = 15\% \text{ to the nearest percent} $$

Notice that even though Examples N and O are similar, the probability for Example O is 15 times greater than the probability from Example N.

CHAPTER 8 REVIEW

5. Multiplication Principle, Permutations and Combinations

a. **Multiplication Principle** If event A can occur in m ways and then event B can occur in n ways, no matter what happens in event A, then event A followed by event B can occur in m × n ways.
b. The product of the whole numbers from 1 through n is written as n! and called **n factorial**.
c. A **permutation of objects** is an arrangement of these objects into a particular order.
d. The number of permutations of ***n*** objects taken ***r*** objects at a time, where $0 \le r \le n$, is

$${}_nP_r = \frac{n!}{(n-r)!}$$

e. A collection of objects for which order is not important is called a **combination**.
f. The number of combinations of ***n*** objects taken ***r*** objects at a time, where $0 \le r \le n$, is

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

CHAPTER TEST

16. A family of mother, father, older sister, brother and younger sister will be randomly assigned seats A, B, C, D and E, in a row, for their flight to Chicago. Seat A is next to the window.
   a. In how many different ways can the family be assigned seats?
   b. In how many different ways can the family be assigned seats, if the older sister is assigned the seat next to the window?
   c. What is the probability that the older sister will be assigned the seat next to the window?
   d. What is the probability that the older sister will not be assigned the seat next to the window?

17. A jar contains 45 balls numbered 1 to 45.
   a. How different sets of five balls can be randomly taken from the jar?
   b. How different sets of five balls containing the ball numbered 42 can be taken from the jar?
   c. What is the probability that a five ball set will contain the ball numbered 42?
   d. What is the probability that a five ball set will not contain the ball numbered 42?

Permutations and Combinations

Calculate each answer in exercises 48 through 51.

48. a. 12!/9! b. 10!/7! 3!

49. a. 15!/13! b. 9!/4! 5!

50. a. ${}_{12}C_4$ b. ${}_{12}P_4$

51. a. ${}_{12}C_8$ b. ${}_{8}P_3$

52. A popular coffee shop has 12 flavors that can be added to a coffee latte. If you chose two flavors with each latte you ordered, how many different two-flavor latte drinks can you order?

53. Twelve students attend a meeting for the school play; in how many ways can 4 students be selected for the parts of chauffer, teacher, coach and parent?

54. An ice cream shop advertises 21 different flavors; how many different 3-scoop dishes of ice cream can you order?

55. For the all-state cross-country meet, the coach will select 4 of the 11 top runners for the 4-person relay race. If the runners are assigned the positions of starter, second runner, third runner and finisher, in how many ways can the relay team be selected?

In a standard deck of 52 playing cards there are 4 jacks, 4 queens and 4 kings, called face cards. Assume that being dealt a hand in cards is like selecting those cards at random from the deck. Use this information in questions 56 through 58.

56. Four card hands
   a. How many different 4-card hands are possible from a deck of 52 cards?
   b. How many different 4-card hands with 4 face cards are possible from a deck of 52 cards?
   c. What is the probability, to four decimal places, of being dealt a 4-card hand of all face cards from a deck of 52 cards?
   d. What is the probability, to four decimal places, of being dealt a 4-card hand of all kings from a deck of 52 cards?

57. Five card hands
   a. How many different 5-card hands are possible from a deck of 52 cards?
   b. How many different 5-card hands with 5 face cards are possible from a deck of 52 cards?
   c. What is the probability, to four decimal places, of being dealt a 5-card hand with all face cards from a deck of 52 cards?
   d. What is the probability, to five decimal places, of being dealt a 5-card hand with no face cards from a deck of 52 cards?

58. Write an explanation for each of the following:

a. Why the probability, to four decimal places, of being dealt a 5-card hand with exactly one ace and four kings is $4/_{52}C_5$.
b. Why the probability, to four decimal places, of being dealt a 5-card hand with four aces is $48/_{52}C_5$.

The names of 10 fifth graders, including the top math and the top spelling student in the fifth grade class are placed in a hat and randomly selected to sit on the stage with the governor for his visit to the school. Use this information in questions 59 and 60.

59. Four students randomly selected to join the governor

a. In how many ways can four students be selected for First Chair, Second Chair, Third Chair and Fourth Chair from the governor for this occasion?
b. In how many ways can three more students be selected for the remaining chairs, if the top math student is to sit in the First Chair?
c. What is the probability that the top math student will sit in the First Chair?

60. Five students randomly selected to join the governor

a. In how many ways can 5 students be selected for First Chair, Second Chair, Third Chair, Fourth Chair and Fifth Chair from the governor for this occasion?
b. In how many ways can three more students be selected for the Third, Fourth and Fifth Chairs, if the top math student is to sit in the First Chair and a different student who is the top spelling student is to sit in the Second Chair?
c. What is the probability, to two decimal places, that the top math student will sit in the First Chair and the top spelling student will sit in the Second Chair?