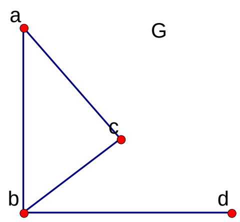
Introduction to Graph Theory

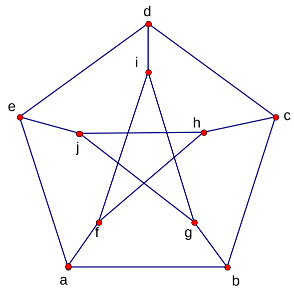
A **graph** G with n **vertices** and m **edges** consists of a vertex set $V(G) =\{v_1,\ldots,v_n\}$ and an edge set $E(G) = \{e_1,\ldots,e_m\}$, where each edge connects exactly two vertices. We may write $G(V,E)$ for the graph whose vertex set is V and edge set is E.

In a **simple graph**, each pair of points is connected by <u>at most one</u> edge.
In a **multigraph**, a pair of points may be connected by more than one edge.

We will write uv for an edge or $\{u,v\}$. We will also say u is **adjacent** to v if $\{u,v\}$ is an edge.

Below are some examples of graphs:





Petersen Graph

The **degree** of a vertex v, written $d(v)$, is the number of edges coming out a vertex. For example, in the above graph G: d(a) = 2; d(b) = 3; d(d) = 1.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write $H \subseteq G$ and say "H is a subgraph of G."
Examples:

An **induced subgraph** G[S] of a graph G is a graph on a vertex subset $S \subseteq V(G)$ such that if $\{u,v\} \in E(G)$ and $u,v \in S$, then $\{u,v\} \in E(G[S])$.
Examples:

A **walk of length** k is a sequence $v_0, e_1, v_1, e_1,\ldots,e_k, v_k$ such that $e_i = v_{i-1}v_i$ for all i.
A **trail** is a walk with no repeated edges.
A **path** is a walk with no repeated vertex.
A **cycle** is a closed trail of length at least one in which the first vertex = last vertex.
e.g.

A **complete graph** on *n* vertices, written K_n, is a graph in which every pair of vertices forms an edge.
Examples:

An **independent set** in a graph G is a vertex subset $S \subseteq V(G)$ such that the induced subgraph G[S] has no edges.

Examples:

A graph is **bipartite** if its vertex set can be partitioned into two independent sets.
e.g.

A **complete bipartite graph** is a bipartite graph in which the edge set consists of all pairs having a vertex from each of the two independent set.
e.g.

Two graphs G&H are **isomorphic** if you can move around the vertices (without removing any edges) of G to make it look like H.
e.g. non e.g.

QUESTIONS

1. How many edges are there in K_n?

2. In a simple graph G on n vertices, what are the possible values for d(v), where v is a vertex of G?

3. Prove the following statement: Every simple graph on at least two vertices has two vertices of equal degree.

4. If G has a vertex set V = {v_1, v_2, …, v_n} and E = {e_1, e_2, …, e_m}, what is $d(v_1) + d(v_2) + ... + d(v_n)$ compared to the number of edges?

5. Can a graph have an odd number of odd degree vertices?

6. How many non-isomorphic graphs can be formed on four vertices?

7. Can a bipartite graph have an odd cycle? Why or why not?

8. A graph is **k-regular** if each vertex has degree k. Show that the number of vertices in a *k*-regular graph is even if *k* is odd.

HW Day 7 (Due 7/5)

1. Find the size of $K_{m,n}$.

2. We write G = (X,Y,E) if G is a bipartite graph, where one of the vertex sets is X; the other is Y and the edge set is E. Suppose G is a regular bipartite graph. Show that both X and Y have the same number of vertices, i.e. $|X| = |Y|$.

3. In a class with nine students, each student sends valentine cards to three others. Is it possible that each student receives cards from the same three students to whom he or she sent cards? If so, show an example. If not, then explain why.

4. Determine which pair of the given graphs are isomorphic. Explain.

5. Create two graphs on 5 vertices that are isomorphic. Try to make them not look like they are obviously isomorphic.

6. Create two graphs G and H on 5 vertices that are NOT isomorphic. Each vertex v_G of G should have a matching vertex v_H in H of the same degree. Explain why they are not isomorphic.