Chapter 2

Exploring What It Means to Know and Do Mathematics

No matter how lucidly and patiently teachers explain to their students, they cannot understand for their students.

Schifter and Fosnot (1993, p. 9)

What does it mean to know a mathematics topic? Take division of fractions, for example. If you know this topic well, what do you know? As mentioned in Chapter 1, the answer is more broad than knowing a procedure you may have memorized (invert the second fraction and multiply). Knowing division of fractions means that you can not only think of examples that fit division of fractions, you can also use alternative strategies to solve problems, estimate an answer, or draw a diagram to show what happens when a number is divided by a fraction. Unfortunately, too much mathematics instruction is limited to simple algorithms without allowing students to deeply learn about different topics.

This chapter is about the learning theory of teaching developmentally and the knowledge necessary for students to learn mathematics with understanding. You might consider this chapter the what, why, and how of teaching mathematics. The "how" is addressed first—how should mathematics be experienced by a learner? Second, "why" should mathematics look this way? And, finally, "what" does it mean to understand mathematics?

Before you read about learning theory and knowledge in mathematics, however, it is important for you to have a chance to "do mathematics" in a way that nurtures understanding and builds connections. These experiences will serve as exemplars when we turn to the discussion of learning theory and knowledge.

What Does It Mean to Do Mathematics?

Stop for a moment and write a few sentences about what it means to do and know mathematics, based on your own experiences. Then put your paper aside until you have finished this chapter.

The description of doing mathematics presented here may not match your personal experiences. That's okay! However, it is not okay to be closed off to new ideas that may clash with your perceptions or to refuse to acknowledge that teaching mathematics could be dramatically different than your previous experience.

Mathematics is more than completing sets of exercises or mimicking processes the teacher explains. Doing mathematics means generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see if your answers make sense. Doing mathematics in classrooms should closely model the act of doing mathematics in the real world.

Mathematics Is the Science of Pattern and Order

This wonderfully simple description of mathematics, found in the thought-provoking publication *Everybody Counts* (Mathematical Sciences Education Board, 1989), challenges the popular view of mathematics as a discipline dominated by computation and rules without reasons. Science is a process of figuring out or making sense. Although you may never have thought of it in quite this way, mathematics is a science of concepts and processes that have a pattern of regularity and logical order. Finding and exploring this regularity or order, and then making sense of it, is what doing mathematics is all about.

Even the youngest schoolchildren can and should be involved in the science of pattern and order.

myeducationlab

Go to the Activities and Application section of Chapter 2 of MyEducationLab. Click on Videos and watch the video entitled **"John Van de Walle on Mathematics Is the Science of Pattern and Order"** to see him give his description of mathematics.

Have you ever noticed that 6 + 7 is the same as 5 + 8 and 4 + 9? What is the pattern? What are the relationships? When two odd numbers are multiplied, the result is also odd, but if the same numbers are added or subtracted, the result is even.

In middle school, students graph linear functions (i.e., functions that can be represented as $y = mx + b$). Graphing functions can be narrowly explored by following a set of steps or rules, but understanding why certain forms of equations, situations, or models are growing in a linear manner involves a search for patterns. Discovering what types of real-world relationships are represented by linear graphs is more scientific—and infinitely more valuable—than creating a graph from an equation without connection to the world.

Engaging in the science of pattern and order—doing mathematics—takes time and effort. Consider topics that show up on lists of "basic skills," such as knowing basic facts for addition and multiplication and having efficient methods of computing whole numbers, fractions, and decimals. Studying relationships on the multiplication chart or analyzing patterns in place value (discussed in detail in the related content chapters) helps students understand what they are doing, therefore increasing their accuracy and retention. To master these topics as facts and procedures by memorization alone is no more doing mathematics than playing scales on the piano is making music.

Pause and Reflect

Envision for a moment an elementary or middle school mathematics class where students are doing mathematics as "a study of patterns." What action verbs would students use to describe what they are doing? Make a short list before reading further.

A Classroom Environment for Doing Mathematics

To create a setting where students are doing mathematics means a shift in the tasks given to students and how classrooms are organized for mathematics lessons. Doing mathematics begins with posing worthwhile tasks and then creating a risk-taking environment where students share and defend mathematical ideas.

The Language of Doing Mathematics. Children in traditional mathematics classes often describe mathematics as "work" or "getting answers." They talk about "plussing" and "doing times" (multiplication). In contrast, the following collection of verbs can be found in most of the literature describing the authentic work of doing mathematics, and all are used in *Principles and Standards* (NCTM, 2000):

explore	justify	construct	develop
investigate	represent	verify	describe
conjecture	formulate	explain	use
solve	discover	predict	

These verbs require higher-level thinking and encompass "making sense" and "figuring out." Children engaged in these actions in mathematics classes will be actively thinking about the mathematical ideas that are involved. Contrast these with the verbs that might reflect the traditional mathematics classroom: listen, copy, memorize, drill. These are lower-level thinking activities and do not adequately prepare students for the real act of doing mathematics. Mathematics requires effort and it is important that students, parents, and the community acknowledge and honor the fact that effort is what leads to learning in mathematics (National Mathematics Advisory Panel, 2008). In classrooms pursuing higher-level mathematics activities on a daily basis, the students are getting an empowering message: "You are capable of making sense of this—you are capable of doing mathematics!"

Every idea introduced in the mathematics classroom can and should be understood by every child. There are no exceptions! All children are capable of learning the mathematics we want them to learn. Their learning becomes meaningful when they are taught using the verbs listed here to perform challenging and engaging mathematics.

The Setting for Doing Mathematics. The teacher's role is to create this spirit of inquiry, trust, and expectation. Within that environment, students are invited to do mathematics. You pose problems; students wrestle toward solutions. The focus is on students actively figuring things out by testing ideas, making conjectures, developing reasons, and offering explanations. In *Classroom Discussions*, a teacher resource describing how to implement effective discourse in the classroom, Chapin, O'Conner, and Anderson (2003) write, "When a teacher succeeds in setting up a classroom in which students feel obligated to listen to one another, to make their own contributions clear and comprehensible, and to provide evidence for their claims, that teacher has set in place a powerful context for student learning" (p. 9).

In the classic book *Making Sense* (Hiebert et al., 1997), the authors describe four features of a productive classroom culture for mathematics in which students can learn from each other.

1. *Ideas are the currency of the classroom.* Ideas, expressed by any participant, have the potential to contribute to everyone's learning and consequently warrant respect and response.
2. *Students have autonomy with respect to the methods used to solve problems.* Students must respect the need for everyone to understand their own methods and must recognize that there are often a variety of methods that will lead to a solution.
3. *The classroom culture exhibits an appreciation for mistakes as opportunities to learn.* Mistakes afford opportunities to examine errors in reasoning, and thereby raise everyone's level of analysis. Mistakes are not to be covered up; they are to be used constructively.

4. *The authority for reasonability and correctness lies in the logic and structure of the subject, rather than in the social status of the participants.* The persuasiveness of an explanation or the correctness of a solution depends on the mathematical sense it makes, not on the popularity of the presenter. (pp. 9–10)

In classrooms that embrace this culture for learning, the way students think about mathematics changes. Rather than students asking (or thinking) "What do you want me to do?" problem ownership shifts the situation to "I think I am going to . . ." (Baker & Baker, 1990). In the latter example the student feels empowered to come up with his or her own approach rather than depend on the teacher to offer an approach. This is foundational in creating an environment for doing mathematics. More information on creating a community of learners is found in Chapter 3.

An Invitation to Do Mathematics

If your goal is to create a classroom environment where children are truly doing mathematics, it is important that you have a personal feel for doing mathematics. The purpose of this section is to provide *you* with opportunities to engage in the science of pattern and order—to do some mathematics. If possible, find one or two peers to work with you so that you can experience sharing and exchanging ideas.

Don't read too much at once. Some hints and suggestions follow each task. Do as much as you can until you are stuck—really stuck—and then read a bit more.

Let's Do Some Mathematics!

We will explore four different problems. Each is independent of the others. None requires any sophisticated mathematics, not even algebra. But they do require higher-level thinking and reasoning. Try out your ideas! Devote time and effort—persist—these are the keys for being successful at mathematics. Have fun!

Start and Jump Numbers: Searching for Patterns

You will need to make a list of numbers that begin with a "start number" and increase by a fixed amount we will call the "jump number." First try 3 as the start number and 5 as the jump number. Write the start number at the top of your list, then 8, 13, and so on, "jumping" by 5 each time until your list extends to about 130.

Examine this list of numbers and find as many patterns as you can. Share your ideas with the group, and write down every pattern you agree really is a pattern.

STOP

Do not read on until you have listed as many patterns as you can identify.

A Few Ideas. Here are some patterns you might consider:

- Do you see at least one alternating pattern?
- Have you looked at odd and even numbers?
- What can you say about the number in the tens place?
- How did you think about the first two numbers with no tens-place digits?
- Have you tried doing any adding of numbers? Numbers in the list? Digits in the numbers?

STOP

If there is an idea in this list you haven't tried, try that now.

Don't forget to think about what happens to your patterns after the numbers go over 100. How are you thinking about 113? One way is as 1 hundred, 1 ten, and 3 ones. But, of course, it could also be "eleventy-three," where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

When you added the digits in the numbers, the sums are 3, 8, 4, 9, 5, 10, 6, 11, 7, 12, 8, ... Did you look at every other number in this string? And what is the sum of the digits for 113? Is it 5 or is it 14? (There is no "right" answer here. But it is interesting to consider different possibilities.)

Next Steps. Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem?

STOP

Try some ideas now before going on.

Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and examine different jump numbers. You will find out that changing jump numbers really "messes things up" a lot compared to changing the start numbers.
- If you have patterns for several different jump numbers, what can you figure out about how a jump number

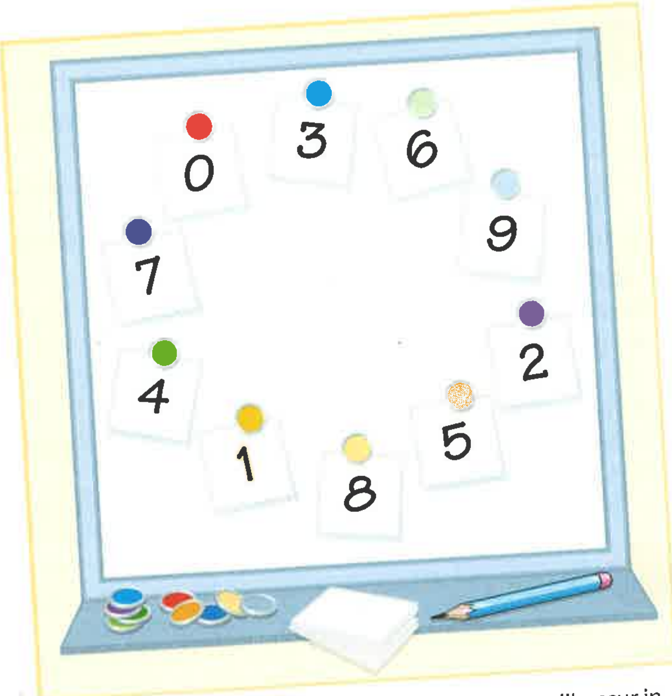


Figure 2.1 For jumps of 3, this cycle of digits will occur in the ones place. The start number determines where the cycle begins.

affects the patterns? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers—it had a "pattern length" of two. But when the jump number is 3, the length of the ones-digit pattern is ten! Do other jump numbers create different pattern lengths?

- For a jump number of 3, how is the ones-digit pattern related to the circle of numbers in Figure 2.1? Are there other circles of numbers for other jump numbers?
- Using the circle of numbers for 3, find the pattern for jumps of multiples of 3, that is, jumps of 6, 9, or 12.

Using Technology. You may want to explore this problem using a calculator, which can make the list generation accessible for young children who can't skip count yet and it opens the door for students to work with bigger jump numbers, such as 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press 3 [+] 5 [=] and then keep pressing [=], the calculator will count by 5s (the first sequence of numbers you wrote). This also works for the other three operations.

Two Machines, One Job

Ron's Recycle Shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

STOP

Do not read on until you either get an answer or get stuck. Can you check that you are correct? Are you sure you are stuck?

A Few Ideas. Are you overlooking any assumptions made in the problem? Do the machines run simultaneously? The problem says "at the same time." Do they run just as fast when working together as when they work alone?

STOP

If this gives you an idea, pursue it before reading more.

Have you tried to predict approximately how much time you think it should take the two machines? Just make an estimate in round numbers. For example, will it be closer to 1 hour or closer to 4 hours? What causes you to answer as you have? Can you tell if your "guestimate" makes sense or is at least in the ballpark? Checking a guess in this way sometimes leads to a new insight.

Some people draw pictures to solve problems. Others like to use something they can move or change. For example, you might draw a rectangle or a line segment to stand for the truckload of paper, or you might get some counters (chips, plastic cubes, pennies) and make a collection that stands for the truckload.

STOP

Go back and work on the problem more.

Consider Solutions of Others. Here are solutions of teachers who worked on this problem (adapted from Schifter & Fosnot, 1993, pp. 24–27). Here is Betsy's solution (she teaches sixth grade):

> Betsy holds up a bar of plastic cubes. "Let's say these 16 cubes are the truckload of paper. In 1 hour, the new machine shreds 8 cubes and the old machine 4 cubes." Betsy breaks off 8 cubes and then 4 cubes. "That leaves these 4 cubes. If the new machine did 8 cubes' worth in 1 hour, it can do 2 cubes' worth in 15 minutes. The old machine does half as much, or 1 cube." As she says this, she breaks off 3 more cubes. "That is 1 hour and 15 minutes, and we still have 1 cube left." Long pause. "Well, the new machine did 2 cubes in 15 minutes, so it will do this cube in $7\frac{1}{2}$ minutes. Add that onto the 1 hour and 15 minutes. The total time will be 1 hour $22\frac{1}{2}$ minutes." (See Figure 2.2.)

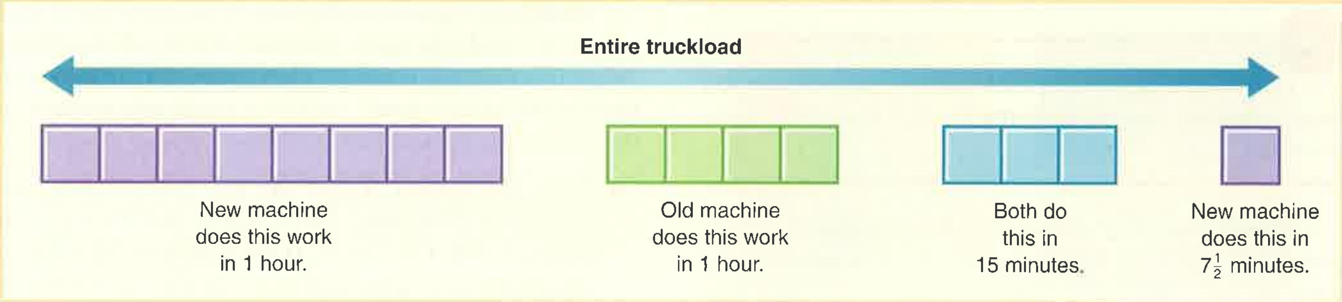


Figure 2.2 Betsy's solution to the paper-shredding problem.

Cora, a fourth-grade teacher, disagrees with Betsy's answer. Here is Cora's proposal:

> "This rectangle [see Figure 2.3] stands for the whole truckload. In 1 hour, the new machine will do half of this." The rectangle is divided in half. "In 1 hour, the old machine could do $\frac{1}{4}$ of the paper." The rectangle is divided accordingly. "So in 1 hour, the two machines have done $\frac{3}{4}$ of the truck, and there is $\frac{1}{4}$ left. What is left is one-third as much as what they already did, so it should take the two machines one-third as long to do that part as it took to do the first part. One-third of an hour is 20 minutes. That means it takes 1 hour and 20 minutes to do it all."

Sylvia, a third-grade teacher, reports on her group's strategy:

> At first, we solved the problem by averaging. We decided that it would take 3 hours because that's the average. Then Deborah asked how we knew to average. We thought we had a reason, but then Deborah asked how Ron would feel if his two machines together took longer than just the new one that could do the job in only 2 hours. So we can see that 3 hours doesn't make sense. So we still don't know whether it's 1 hour and 20 minutes or 1 hour and $22\frac{1}{2}$ minutes.

As with the teachers in these examples, it is important to decide if your solution is correct through justifying why you did what you did, as this reflects real problem solving (rather than checking with an answer key). After you have justified that you have solved the problem in a correct manner, try to find other ways to reach that solution or try to understand others' approaches to the problem—in considering other ways, you can expand your repertoire of problem-solving strategies.

One Up, One Down

For Grades 1–3. **When you add 7 to itself, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:**

↑ ↓
7 + 7 = 14 has the same answer as 8 + 6 = 14

It works for 5 + 5 too:

↑ ↓
5 + 5 = 10 has the same answer as 6 + 4 = 10

What can you find out about this?

For Grades 4–8. **What happens when you change addition to multiplication in this exploration?**

↑ ↓
7 × 7 = 49 has an answer that is one more than 8 × 6 = 48

It works for 5 × 5 too:

↑ ↓
5 × 5 = 25 has an answer that is one more than 6 × 4 = 24

What can you find out about this situation?

Can this pattern be extended to other situations?

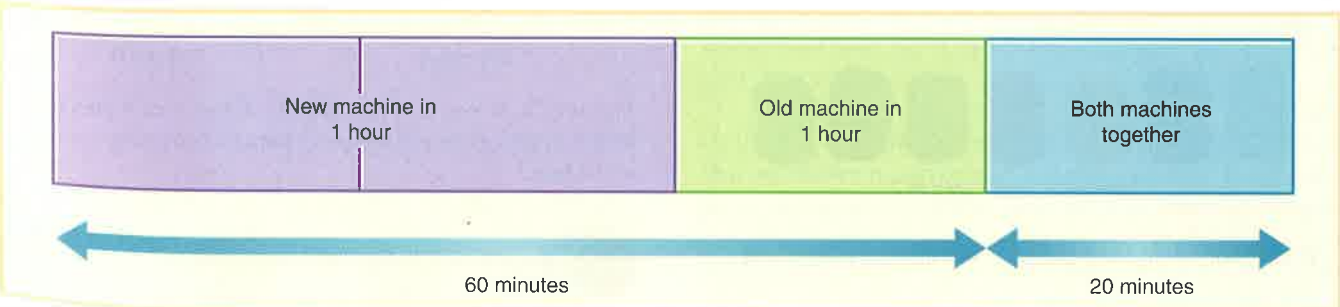


Figure 2.3 Cora's solution to the paper-shredding problem.

STOP

Work on the multiplication pattern. Explore until you have developed some ideas. Write down whatever ideas you discover.

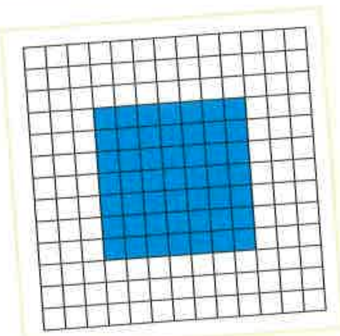
A Few Ideas. Use a physical model or picture. You have probably found some interesting patterns. Can you tell why these patterns work? In the case of addition, it is fairly easy to see that when you take from one number and give to the other, the total stays the same. With multiplication, that is not the case. Why? One way to explore this is to draw rectangles with a length and height of each of the factors (e.g., for the first problem, a 7-by-7-unit rectangle and a 6-by-4-unit rectangle). See how the rectangles compare (Figure 2.4(a)).

You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7 × 7 is seven stacks with seven chips in each stack (set). The expression 8 × 6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare (Figure 2.4(b)).

STOP

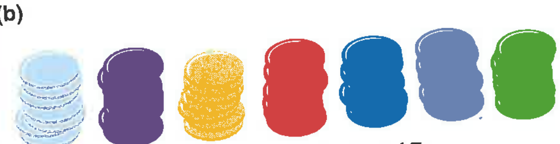
Work with one or both of these approaches to see if you get any insights.

(a)



This is 7 × 7 shown as an array of 7 rows of 7.

(b)



This is 7 × 7 as 7 sets of 7.

What happens when you change one of these to show 6 × 8?

Figure 2.4 Two physical ways to think about multiplication that might help in the exploration.

Additional Patterns to Explore. There is a lot to find out about multiplication patterns. Think of the many "what if"s that are possible. Here are a few. If you have found other ones—great. There are many ways to explore this problem.

- Have you looked at how the first two numbers are related? For example, 7 × 7, 5 × 5, and 9 × 9 are all products with like factors. What if the product was two consecutive numbers (e.g., 8 × 7 or 13 × 12)? What if the factors differ by 2 or by 3?
- Think about adjusting by numbers other than one. What if you adjust up two and down two (e.g., 7 × 7 to 9 × 5)?
- What happens if you use big numbers instead of small ones (e.g., 30 × 30)?
- If both factors increase, is there a pattern?

We hope you have made your own conjectures and explored them or at least added to the "what if" list. Scientists (including mathematicians) explore new ideas that strike them as interesting and promising rather than blindly following procedures.

The Best Chance of Purple

Three students are spinning to "get purple" with two spinners, either by spinning first red and then blue or first blue and then red (see Figure 2.5). They may choose to spin each spinner once or one of the spinners twice. Mary chooses to spin twice on spinner A; John chooses to spin twice on spinner B; and Susan chooses to spin first on spinner A and then on spinner B. Who has the best chance of getting a red and a blue? (Lappan & Even, 1989, p. 17)

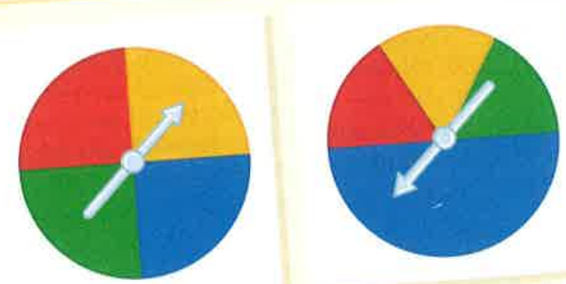


Figure 2.5 You may spin A twice, B twice, or A then B. Which option gives you the best chance of spinning a red and a blue?

STOP

Think about the problem and what you know. Experiment.

A Few Ideas. Sometimes it is tough to get a feel for problems that are abstract or complex. In situations involving chance, find a way to experiment and see what happens. For this problem, you can make spinners using a freehand drawing on paper, a paper clip, and a pencil. Put your pencil point through the loop of the clip and on the center of your spinner. Now you can spin the paper clip "pointer." Try at least 20 pairs of spins for each choice and keep track of what happens.

Consider these issues as you explore:

- For Susan's choice (A then B), would it matter if she spun B first and then A? Why or why not?
- Explain why you think purple is more or less likely in one of the three cases compared to the other two. It sometimes helps to talk through what you have observed to come up with a way to apply some more precise reasoning.

STOP

Try these suggestions before reading on.

Strategy 1: Tree Diagrams. On spinner A, the four colors each have the same chance of coming up. You could make a tree diagram for A with four branches, and all the branches would have the same chance (see Figure 2.6). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?
- How could you make a tree diagram for B with each branch having the same chance?
- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?
- Which branches on your diagram represent getting purple?
- How could you make tree diagrams for John's and Susan's choices? Why do they make sense?

Test your ideas by actually spinning the spinner or spinners.

Tree diagrams are only one way to approach this. You may have a different way. As long as your way seems to be getting you somewhere, stick with it. There is one more suggestion to follow, but don't read further if you are ready to solve the problem.

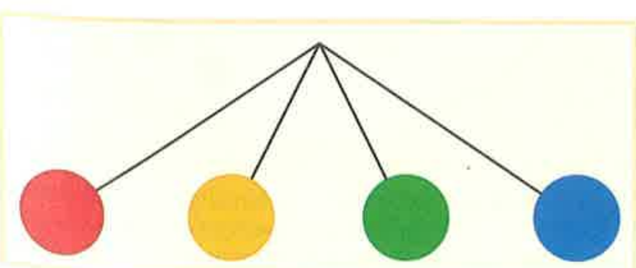


Figure 2.6 A tree diagram for spinner A in Figure 2.5.

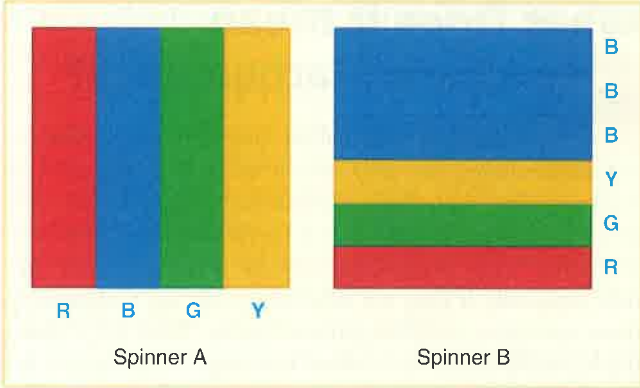


Figure 2.7 A square shows the chance of obtaining each color for the spinners in Figure 2.5.

Strategy 2: Grids. Suppose that you had a square that represented all the possible outcomes for spinner A and a similar square for spinner B. Although there are many ways to divide a square in four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). When the second event (here the second spin) follows the first event, make the lines on the second square go the opposite way from the lines on the first. Make a tracing of one square in Figure 2.7 and place it on the other. You end up with 24 little sections.

Why are there six subdivisions for the spinner B square? What does each of the 24 little rectangles stand for? What sections would represent purple? In any other method you have been trying, did 24 come into play when you were looking at spinner A followed by B?

Where Are the Answers?

No answers or solutions are given in this text. How do you feel about that? What about the "right" answers? Are your answers correct? What makes the solution to any investigation "correct"?

In the classroom, the ready availability of the answer book or the teacher's providing the solution or verifying that an answer is correct sends a clear message to students about doing mathematics: "Your job is to find the answers that the teacher already has." In the real world of problem solving outside the classroom, there are no teachers with answers and no answer books. Doing mathematics includes using justification as a means of determining if an answer is correct.

What Does It Mean to Learn Mathematics?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Can students solve such challenging tasks? Why take the time to solve these problems—isn't it better to do a lot of shorter problems? Why should students be doing problems like this, especially if they are reluctant to do so? Collectively, these questions could be summarized as "How does 'doing mathematics' relate to student learning?" The answer lies in current theory and research on how people learn, as discussed in the following sections. The experiences we provide in classrooms should be designed to maximize learning opportunities for students.

Constructivist Theory

Constructivism is rooted in the cognitive school of psychology and in the work of Jean Piaget, who introduced the notion of mental schema and developed a theory of cognitive development in the 1930s (translated to English in the 1950s). At the heart of constructivism is the notion that children (or any learners) are not blank slates but rather creators of their own learning. Integrated networks, or *cognitive schemas*, are both the product of constructing knowledge and the tools with which additional new knowledge can be constructed. As learning occurs, the networks are rearranged, added to, or otherwise modified. Piaget suggested that schemas can be changed in two ways—*assimilation* and *accommodation*. Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing network. Accommodation takes place when the new concept does not "fit" with the existing network, so the brain revamps or replaces the existing schema. Through *reflective thought*, people modify their existing schemas to incorporate new ideas (Fosnot, 1996). Reflective thought means sifting through existing ideas (also called prior knowledge) to find those that seem to be related to the current thought, idea, or task.

Existing schemas are often referred to as prior knowledge. One basic tenet of constructivism is that people construct their own knowledge based on their prior knowledge. All people, all of the time, construct or give meaning to things they perceive or think about. As you read these words, you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge to make sense of the new information.

Construction of Ideas. To construct or build something in the physical world requires tools, materials, and effort. How we construct ideas can be viewed in an analogous manner. The tools we use to build understanding are our existing ideas and knowledge. The materials we use to build understanding may be things we see, hear, or touch—elements of our physical surroundings. Sometimes the materials are our own thoughts and ideas. The effort required is active and reflective thought.

In Figure 2.8 blue and red dots are used as symbols for ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas (blue dots) are used in the construction will necessarily be connected to the new idea (red dot) because those were the ideas that gave meaning to it. If a potentially relevant idea (blue dot) is not accessed by the learner when learning a new concept (red dot), then that potential connection will not be made.

Constructing knowledge is an active endeavor on the part of the learner (Baroody, 1987; Cobb, 1988; Fosnot, 1996; von Glasersfeld, 1990, 1996). To construct and understand a new idea requires actively thinking about it. "How does this fit with what I already know?" "How can I understand this in the context of my current understanding of this idea?" Knowledge cannot be "poured into" a learner. Put simply, constructing knowledge requires *reflective thought*, actively thinking about or mentally working on an idea.

Learners will vary in the number and nature of connections they make between a new idea and existing ideas.

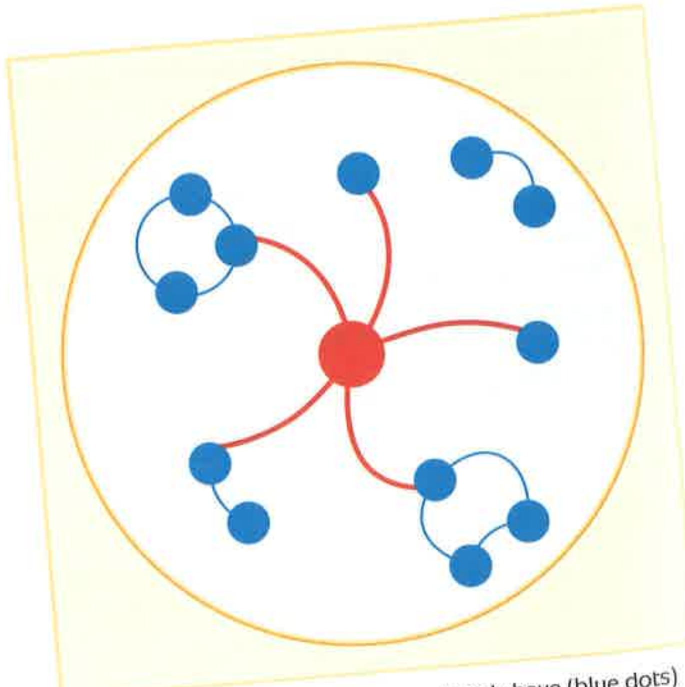


Figure 2.8 We use the ideas we already have (blue dots) to construct a new idea (red dot), developing in the process a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

The construction of an idea is going to be different for each learner, even within the same environment or classroom. Though learning is constructed within the self, the classroom culture contributes to learning while the learner contributes to the culture in the classroom (Yackel & Cobb, 1996). Yackel and Cobb argue that the learner and the culture of the classroom are reflexively related—one influencing the other.

Sociocultural Theory

In the same way that the work of Piaget led to constructivism, the work of Lev Vygotsky, a Russian psychologist, has greatly influenced sociocultural theory. Vygotsky's work also emerged in the 1920s and 1930s, though it was not translated until the late 1970s. There are many concepts that these theories share (for example the learning process as active meaning-seeking on the part of the learner), but sociocultural theory has several unique foundational concepts. One is that mental processes exist between and among people in social learning settings, and that from these social settings the learner moves ideas into his or her own psychological realm (Forman, 2003).

Second, the way in which information is internalized (or learned) depends on whether it was within a learner's zone of proximal development (ZPD), which is the difference between a learner's assisted and unassisted performance on a task (Vygotsky, 1978). Simply put, the ZPD refers to a "range" of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support of peers or more knowledgeable others. "[T]he ZPD is not a physical space, but a symbolic space created through the interaction of learners with more knowledgeable others and the culture that precedes them" (Goos, 2004, p. 262). Both Cobb (1994) and Goos (2004) suggest that in a true mathematical community of learners there is something of a common ZPD that emerges across learners as well as the ZPDs of individual learners.

Another major concept in sociocultural theory is *semiotic mediation*, a term used to describe how information moves from the social plane to the individual plane. It is defined as the "mechanism by which individual beliefs, attitudes, and goals are simultaneously affected and affect sociocultural practices and institutions" (Forman & McPhail, 1993, p. 134). Semiotic mediation involves interaction through language but also through diagrams, pictures, and actions. Language and these other objects and actions are considered the "tools" of mediation.

Social interaction is essential for mediation. The nature of the community of learners is affected by not just the culture the teacher creates, but the broader social and historical culture of the members of the classroom (Forman, 2003). In summary, from a sociocultural perspective, learning is dependent on the learners (working within their ZPD), the social interactions in the classroom, and the culture within and beyond the classroom.

Implications for Teaching Mathematics

It is not necessary to choose between a social constructivist theory that favors the views of Vygotsky and a cognitive constructivism built on the theories of Piaget (Cobb, 1994). In fact, when considering classroom practices that maximize opportunities to construct ideas, or to provide tools to promote mediation, they are quite similar. Classroom discussion based on students' own ideas and solutions to problems is absolutely "foundational to children's learning" (Wood & Turner-Vorbeck, 2001, p. 186).

It is important to restate that a learning theory is not a teaching strategy, but the theory *informs* teaching. In this section teaching strategies that reflect constructivist and sociocultural perspectives are briefly discussed. You will see these strategies revisited in Chapters 3 and 4, where a problem-based model for instruction is discussed, and throughout the content chapters, where you learn how to apply these ideas to specific areas of mathematics.

Build New Knowledge from Prior Knowledge. Consider the following task, posed to a class of fourth graders who are learning division of whole numbers.

> Four children had 3 bags of M&Ms. They decided to open all 3 bags of candy and share the M&Ms fairly. There were 52 M&M candies in each bag. How many M&M candies did each child get? (Campbell & Johnson, 1995, pp. 35–36)

Note: You may want to select a nonfood context, such as decks of cards, or any culturally relevant or interesting item that would come in similar quantities.

STOP

Consider how you might introduce division to fourth graders and what your expectations might be for this problem as a teacher grounding your work in constructivist or sociocultural learning theory.

The student work samples in Figure 2.9 are from a classroom that is grounded in constructivist ideas—that students should develop, or invent, strategies for doing mathematics using their prior knowledge, therefore making connections among mathematics concepts.

Marlena interpreted the task as "How many sets of 4 can be made from 156?" She first used facts that were either easy or available to her: 10×4 and 4×4. These totals she subtracted from 156 until she got to 100. This seemed to cue her to use 25 fours. She added her sets of 4 and knew the answer was 39 candies for each child. Marlena is using an equal subtraction approach and known multiplication facts. Note the "blue dots" that she is connecting in order to begin learning about the newer concept of division. While this is not the most efficient approach, it demonstrates that

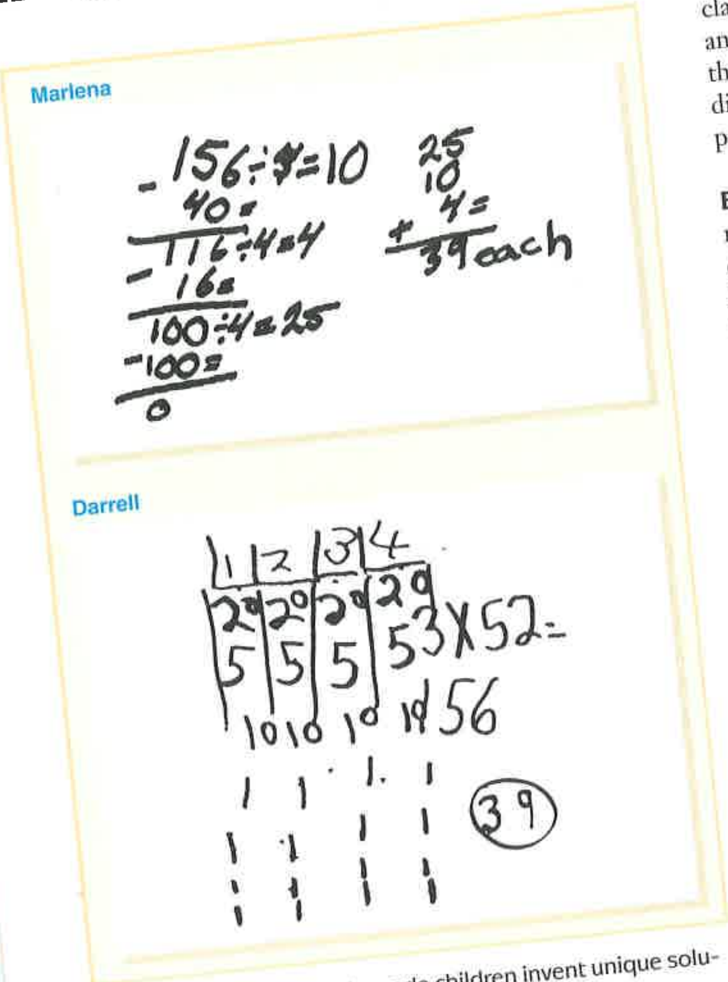


Figure 2.9 Two fourth-grade children invent unique solutions to a computation.

Source: Reprinted with permission from P. F. Campbell and M. L. Johnson, "How Primary Students Think and Learn," in I. M. Carl (Ed.), *Prospects for School Mathematics* (pp. 21–42), copyright © 1995 by the National Council of Teachers of Mathematics, Inc. www.nctm.org.

Marlena understands the concept of division and can move towards more efficient approaches.

Darrell's approach was more directly related to the sharing context of the problem. He formed four columns and distributed amounts to each, accumulating the amounts mentally and orally as he wrote the numbers. Darrell used a counting-up approach, first giving each student 20 M&Ms, seeing they could get more, distributed 5, then 10, then singles until he reached the total. Like Marlena, Darrell used facts and procedures that he knew. The context of sharing provided a "blue dot" for Darrell, as he was able to think about the problem in terms of equal distribution.

Provide Opportunities to Talk about Mathematics. Learning is enhanced when the learner is engaged with others working on the same ideas. A worthwhile goal is to create an environment in which students interact with each other and with you. The rich interaction in such a classroom allows students to engage in reflective thinking and to internalize concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

Build In Opportunities for Reflective Thought. Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea you are teaching to be interconnected in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.8, we want to activate every blue dot students have that is related to the new red dot we want them to learn.

As you will see in Chapter 3 and throughout this book, a significant key to getting students to be reflective is to engage them in problems where they use their prior knowledge as they search for solutions and create new ideas in the process. The problem-solving approach requires not just answers but also explanations and justifications for solutions.

Encourage Multiple Approaches. Teaching should provide opportunities for students to build connections between what they know and what they are learning. The student whose work is presented in Figure 2.10 may not understand the algorithm she is trying to use. If instead she was asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact, like 7 × 8, can have better results if a teacher promotes multiple strategies. Imagine a class where children discuss and share clever ways to figure out the product. One child might think of 5 eights and then 2 more eights. Another may have learned 7 × 7 and added on 7 more. Still another might take half of the sevens (4 × 7) and double that. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical "dots" relating addition and multiplication concepts.

In contrast, facts such as 7 × 8 can be learned by rote (memorized). While that knowledge is still constructed, it is not connected to other knowledge. Rote learning can be thought of as a "weak construction" (Noddings, 1993). Students can recall it if they remember it, but if they forget, they don't have 7 × 8 connected to other knowledge pieces that would allow them to redetermine the fact.

Treat Errors as Opportunities for Learning. When students make errors, it can mean a misapplication of their prior knowledge in the new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Knowing that children rarely give random

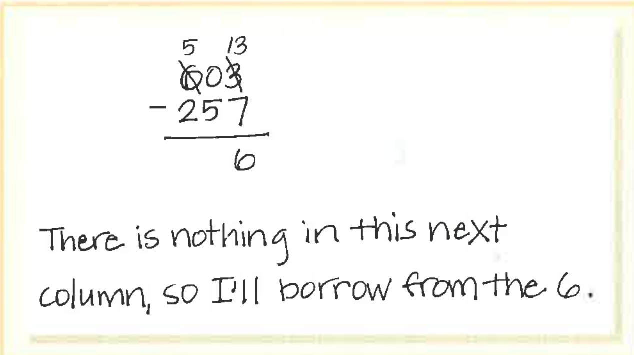


Figure 2.10 This student's work indicates that she has a misconception about place value and regrouping.

responses (Ginsburg, 1977; Labinowicz, 1985) gives insight into addressing student misconceptions and helping students accommodate new learning. For example, students comparing decimals may incorrectly apply "rules" of whole numbers, such as "the longer the number the bigger" (Martinie, 2007; Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989).

Figure 2.10 is an example of a student incorrectly applying what she learned about regrouping. If the teacher tries to help the student by re-explaining the "right" way to do the problem, the student loses the opportunity to reflect on and correct her misconceptions. If the teacher instead asks the student to explain her regrouping process, the student must engage her reflective thought and think about what was regrouped and how to keep the number equivalent.

Scaffold New Content. The concept of *scaffolding*, which comes out of sociocultural theory, is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools like manipulatives or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of "blue dots."

Honor Diversity. Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honoring those experiences. Thus, lessons begin with eliciting prior experiences, and understandings and contexts for the lessons are selected based on students' knowledge and experiences. Some students will not have the "blue dots" they need—it is your job to provide experiences where those blue dots are developed and then connected to the concept being learned.

Classroom culture influences the individual learning of your students. As stated previously, you should support a range of approaches and strategies for doing mathematics. Students' ideas should be valued and included in classroom discussion of the mathematics. This shift in practice, away from the teacher telling one way to do the problem, establishes a classroom culture where ideas are valued. This approach values the uniqueness of each individual.

What Does It Mean to Understand Mathematics?

Both constructivist and sociocultural theories emphasize the learner building connections (blue dots to the red dots) among existing and new ideas. So you might be asking, "What is it they should be learning and connecting?" Or "What are those blue dots?" This section focuses on mathematics content required in today's classrooms.

It is possible to say that we know something or we do not. That is, an idea is something that we either have or don't have. Understanding is another matter. For example, most fifth graders know something about fractions. Given the fraction $\frac{6}{8}$, they likely know how to read the fraction and can identify the 6 and 8 as the numerator and denominator, respectively. They know it is equivalent to $\frac{3}{4}$ and that it is more than $\frac{1}{2}$.

Students will have different *understandings*, however, of such concepts as what it means to be equivalent. They may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not understand that $\frac{3}{4}$ and $\frac{6}{8}$ represent identical quantities. Some may think that simplifying $\frac{6}{8}$ to $\frac{3}{4}$ makes it a smaller number. Some students will be able to create pictures or models to illustrate equivalent fractions or will have many examples of how $\frac{6}{8}$ is used outside of class. In summary, there is a range of ideas that students often connect to their individualized *understanding* of a fraction—each student brings a different set of blue dots to his or her knowledge of what a fraction is.

Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas. Understanding is not an all-or-nothing proposition. It depends on the existence of appropriate ideas and on the creation of new connections, varying with each person (Backhouse, Haggarty, Pirie, & Stratton, 1992; Davis, 1986; Hiebert & Carpenter, 1992; Janvier, 1987; Schroeder & Lester, 1989).

One way that we can think about understanding is that it exists along a continuum from a relational understanding—knowing what to do and why—to an instrumental understanding—doing without understanding (see Figure 2.11). The two ends of this continuum were named by Richard Skemp (1978), an educational psychologist who has had a major influence on mathematics education.

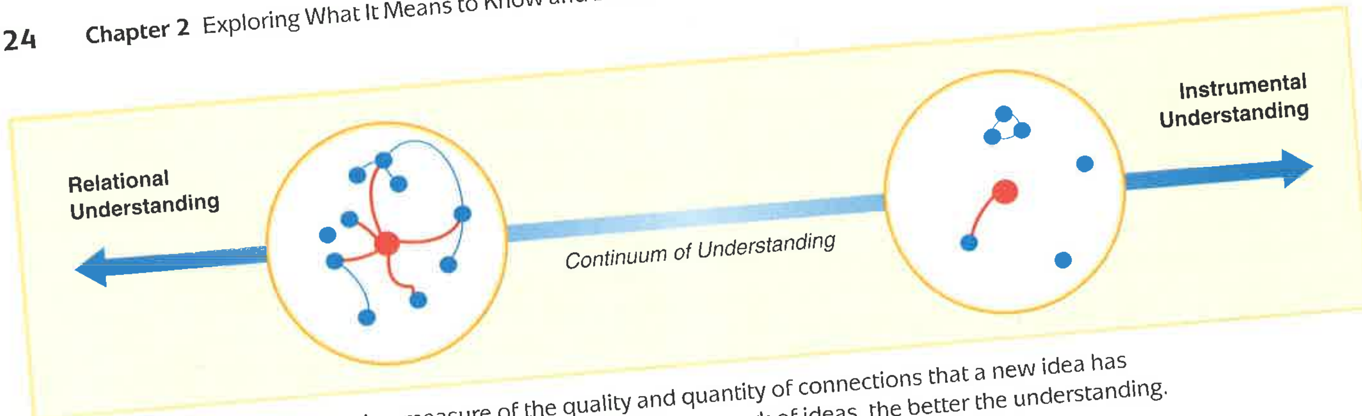


Figure 2.11 Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

In the $\frac{6}{8}$ example, the student who can draw diagrams, give examples, find equivalencies, and approximate the size of $\frac{6}{8}$ has an understanding toward the relational end of the continuum, while a student who only knows the names and a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding more on the instrumental end of the continuum.

Mathematics Proficiency

Much work has emerged since Skemp's classic work on relational and instrumental understanding focusing on what mathematics should be learned, all of it based on the need to include more than learning procedures.

Conceptual and Procedural Understanding. Conceptual understanding is knowledge about the relationships or foundational ideas of a topic. Procedural understanding is knowledge of the rules and procedures used in carrying out mathematical processes and also the symbolism used to represent mathematics. Consider the task of multiplying 47 × 21. The conceptual understanding of this problem includes such ideas as that multiplication is repeated addition and that the problem could represent the area of a rectangle with dimensions of 47 inches and 21 inches. The procedural knowledge could include the standard algorithm or invented algorithms (e.g., multiplying 47 by 10, doubling it, then adding one more 47). The ability to employ invented strategies, such as the one described here, requires a conceptual understanding of place value and multiplication.

In fact, it is well established in research on mathematics learning that conceptual understanding is an important component of procedural proficiency (Bransford, Brown, & Cocking, 2000; National Mathematics Advisory Panel, 2008; NCTM, 2000). The *Principles and Standards* Learning Principle states it well:

"The alliance of factual knowledge, procedural proficiency, and conceptual understanding makes all three components usable in powerful ways" (p. 19). ◆

Recall the two students who used their own invented procedure to solve 156 ÷ 4 (see Figure 2.9). Clearly, there was an active and useful interaction between the procedures the children invented and the concepts they knew about multiplication and were constructing about division.

The common practice of teaching procedures in the absence of conceptual understanding leads to errors and a dislike of mathematics. One way to help your students (and you) think about all the interrelated ideas for a concept is to create a network or web of associations, as demonstrated in Figure 2.12 for the concept of ratio. Note how much is involved in having a relational understanding of ratio. Compare that to the instrumental treatment of ratio in some textbooks that have a single lesson on the topic with prompts such as "If the ratio of girls to boys is 3 to 4, then how many girls are there if there are 24 boys?"

Five Strands of Mathematical Proficiency. While conceptual and procedural understanding of any concept are essential, they are not sufficient. Being mathematically proficient means that people exhibit behaviors and dispositions as they are "doing mathematics." *Adding It Up* (NRC, 2001), an influential report on how students learn mathematics, describes five strands involved in being mathematically proficient: (1) conceptual understanding, (2) procedural fluency, (3) strategic competence, (4) adaptive reasoning, and (5) productive disposition. Figure 2.13 illustrates these interrelated and interwoven strands, providing a definition of each.

Recall the problems that you solved in the "Let's Do Some Mathematics" section. In approaching each problem, if you felt like you could design a strategy to solve it (or try new approaches if one didn't work), then that is evidence of strategic competence. In each of the problems selected, you were asked to explain or justify solutions. If you were able to reason about a pattern and tell how you knew it would work, this is evidence of adaptive reasoning. Finally, if you were committed to making sense of and solving those tasks, knowing that if you kept at it, you would get to a solution, then you have a productive disposition.

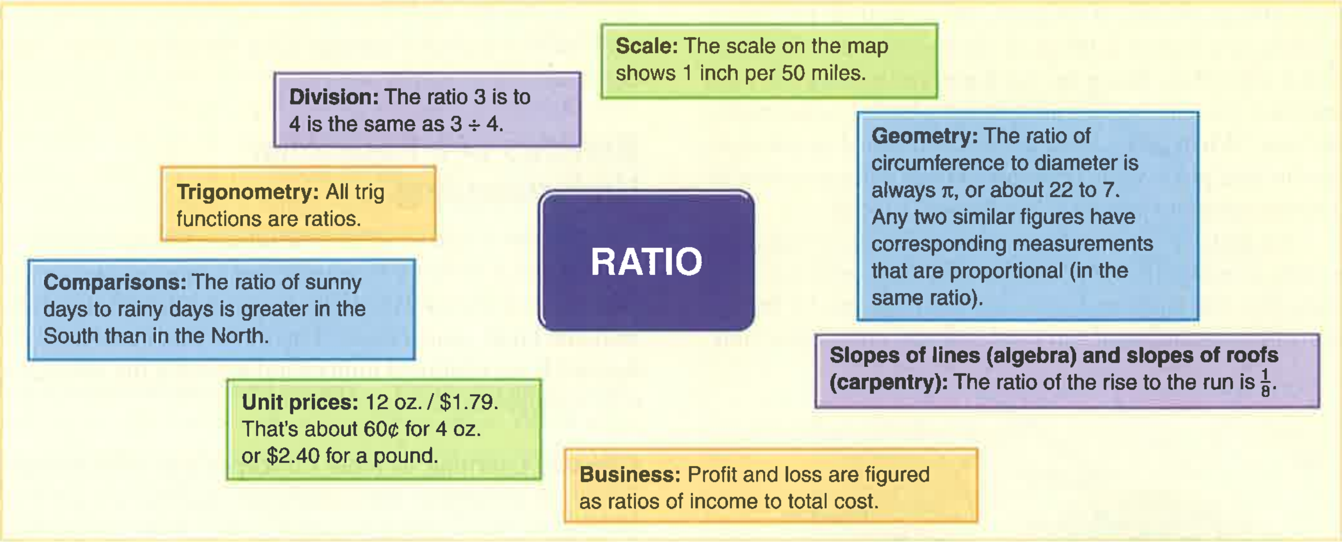


Figure 2.12 Potential web of ideas that could contribute to the understanding of "ratio."

The last three of the five strands develop only when students have experiences that involve these processes. We hope you have noticed that the terms used here are very similar to the ones in the previous learning theory discussion. Reflection, using prior knowledge, social interaction, and solving problems in a variety of ways, among other strategies, are essential to learning and therefore becoming mathematically proficient.

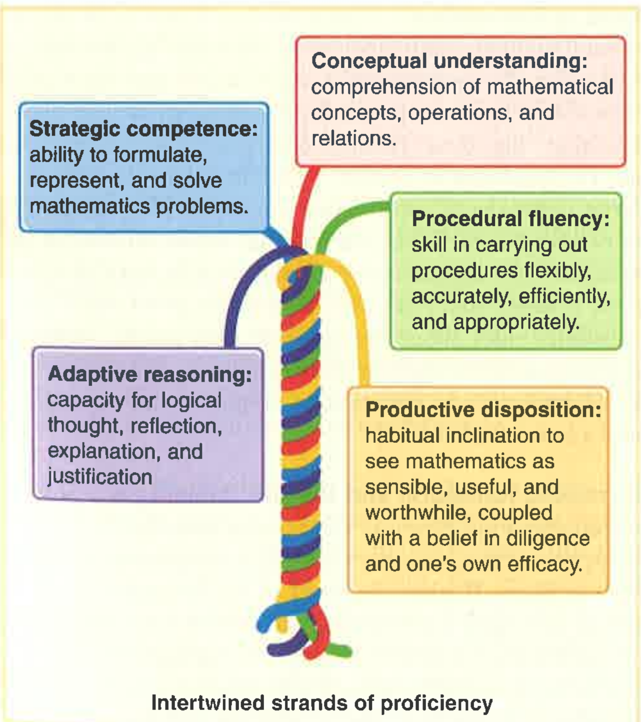


Figure 2.13 *Adding It Up* describes five strands of mathematical proficiency.

Source: Adding It Up: Helping Children Learn Mathematics, p. 5. Reprinted with permission from the National Academies Press, copyright © 2001, National Academy of Sciences.

Implications for Teaching Mathematics

If we accept the notion that understanding has both qualitative and quantitative differences from knowing, the question "Does she know it?" must be replaced with "How does she understand it? What ideas does she connect with it?" In the following examples, you will see how different children may well develop different ideas about the same knowledge and, thus, have different understandings.

Early Number Concepts. Consider the concept of "7" as constructed by a child in the first grade. A first grader most likely connects "7" to the counting procedure and the construct of "more than," probably understanding it as less than 10 and more than 2. What else will this child eventually connect to the concept of 7? It is 1 more than 6; it is 2 less than 9; it is the combination of 3 and 4 or 2 and 5; it is odd; it is small compared to 73 and large compared to $\frac{1}{10}$; it is the number of days in a week; it is "lucky"; it is prime; and on and on. The web of potential ideas connected to a number can grow large and involved.

Computation. Computation is much more than memorizing a procedure; analyzing a student's strategy provides a good opportunity to see how understanding can differ from one child to another. For addition and subtraction with two- or three-digit numbers, a flexible and rich understanding of numbers and place value is very helpful. How might different children approach the task of finding the sum of 37 and 28? For children whose understanding of 37 is based only on counting, the use of counters and a count-all procedure is likely (see Figure 2.14(a)). A student may use the traditional algorithm, lining up the digits and adding the ones and then the tens, but not understand why they are carrying the one. When procedures are not connected to concepts (in this case place-value concepts), errors and unreasonable answers are more common (see Figure 2.14(b)).

Students can solve the problem using an invented approach (see Figure 2.14(c) & (d)). The strategies used here show that the students know that numbers can be broken apart in many different ways and that the sum of two numbers remains the same if you add something to one number and subtract an equal amount from the other. These students can add in *flexible* ways.

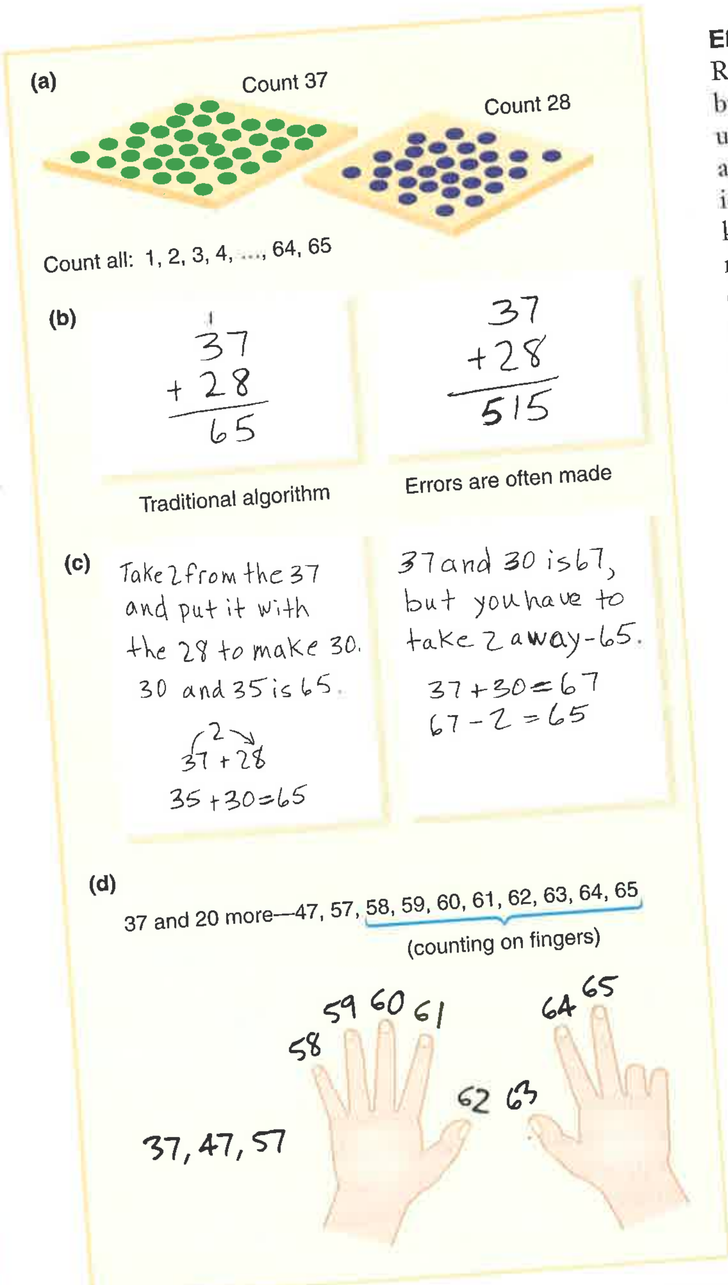


Figure 2.14 A range of computational examples showing different levels of understanding.

Benefits of a Relational Understanding

To teach for a rich or relational understanding requires a lot of work and effort. Concepts and connections develop over time, not in a day. Tasks must be selected that help students build connections. The important benefits to be derived from relational understanding make the effort not only worthwhile but also essential.

Effective Learning of New Concepts and Procedures. Recall what learning theory tells us—students are actively building on their existing knowledge. The more robust their understanding of a concept, the more connections students are building, and the more likely it is they can connect new ideas to the existing conceptual webs they have. Fraction knowledge and place-value knowledge come together to make decimal learning easier, and decimal concepts directly enhance an understanding of percentage concepts and procedures. Without these and many other connections, children will need to learn each new piece of information they encounter as a separate, unrelated idea.

Less to Remember. When students learn in an instrumental manner, mathematics can seem like endless lists of isolated skills, concepts, rules, and symbols that often seem overwhelming to keep straight. Constructivists talk about teaching "big ideas" (Brooks & Brooks, 1993; Hiebert et al., 1996; Schifter & Fosnot, 1993). Big ideas are really just large networks of interrelated concepts. Frequently, the network is so well constructed that whole chunks of information are stored and retrieved as single entities rather than isolated bits. For example, knowledge of place value subsumes rules about lining up decimal points, ordering decimal numbers, moving decimal points to the right or left in decimal-percent conversions, rounding and estimating, and a host of other ideas.

Increased Retention and Recall. Memory is a process of retrieving information. Retrieval of information is more likely when you have the concept connected to an entire web of ideas. If what you need to recall doesn't come to mind, reflecting on ideas that are related can usually lead you to the desired idea eventually. For example, if you forget the formula for surface area of a rectangular solid, reflecting on what it would look like if unfolded and spread out flat can help you remember that there are six rectangular faces in three pairs that are each the same size.

Enhanced Problem-Solving Abilities. The solution of novel problems requires transferring ideas learned in one context to new situations. When concepts are embedded in a rich network, transferability is significantly enhanced and, thus, so is problem solving (Schoenfeld, 1992). When students understand the relationship between a situation and a context, they are going to know when to use a particular approach to solve a problem. While many students may be able to do this with whole-number computation, once problems increase in difficulty and numbers move to rational numbers or unknowns, students without a relational understanding are not able to apply the skills they learned to solve new problems.

Improved Attitudes and Beliefs. Relational understanding has an affective as well as a cognitive effect. When ideas are well understood and make sense, the learner tends to develop a positive self-concept about his or her ability to learn and understand mathematics. There is a definite feeling of "I can do this! I understand!" There is no reason to fear or to be in awe of knowledge learned relationally. At the other end of the continuum, instrumental understanding has the potential of producing mathematics anxiety, a real phenomenon that involves fear and avoidance behavior.

Multiple Representations to Support Relational Understanding

The more ways that children are given to think about and test an emerging idea, the better chance they will correctly form and integrate it into a rich web of concepts and therefore develop a relational understanding. Lesh, Post, and Behr (1987) offer five "representations" for concepts (see Figure 2.15). Their research has found that children who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations. Strengthening the ability to move between and among these representations improves student understanding and retention. Discussion of oral language, real-world situations, and written symbols is woven into this chapter, but it is important that you have a good perspective on how manipulatives and models can help or fail to help children construct ideas.

Models and Manipulatives. A *model for a mathematical concept* refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed. In this sense, any group of 100 objects can be a model of the concept "hundred" because we can impose the 100-to-1 relationship on the group and a single element of the group. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics, like connecting cubes, or objects that were created for other purposes.

It is incorrect to say that a model "illustrates" a concept. To illustrate implies showing. Technically, all that you actually see with your eyes is the physical object; only your mind can impose the mathematical relationship on the object (Suh, 2007; Thompson, 1994).

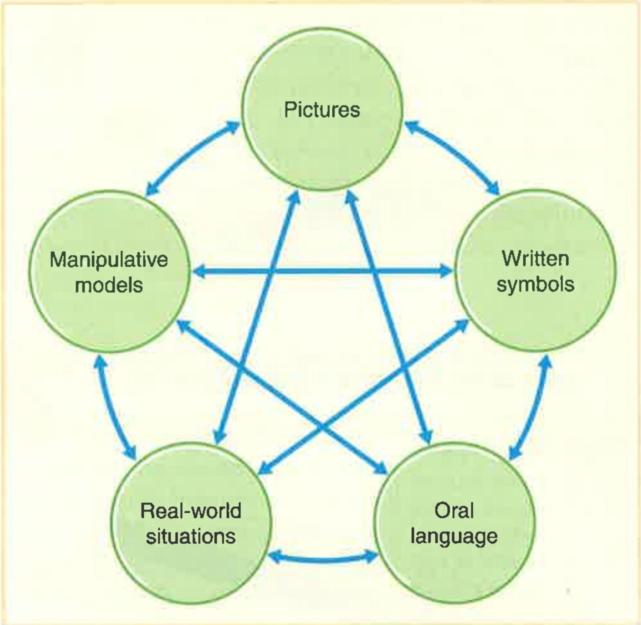


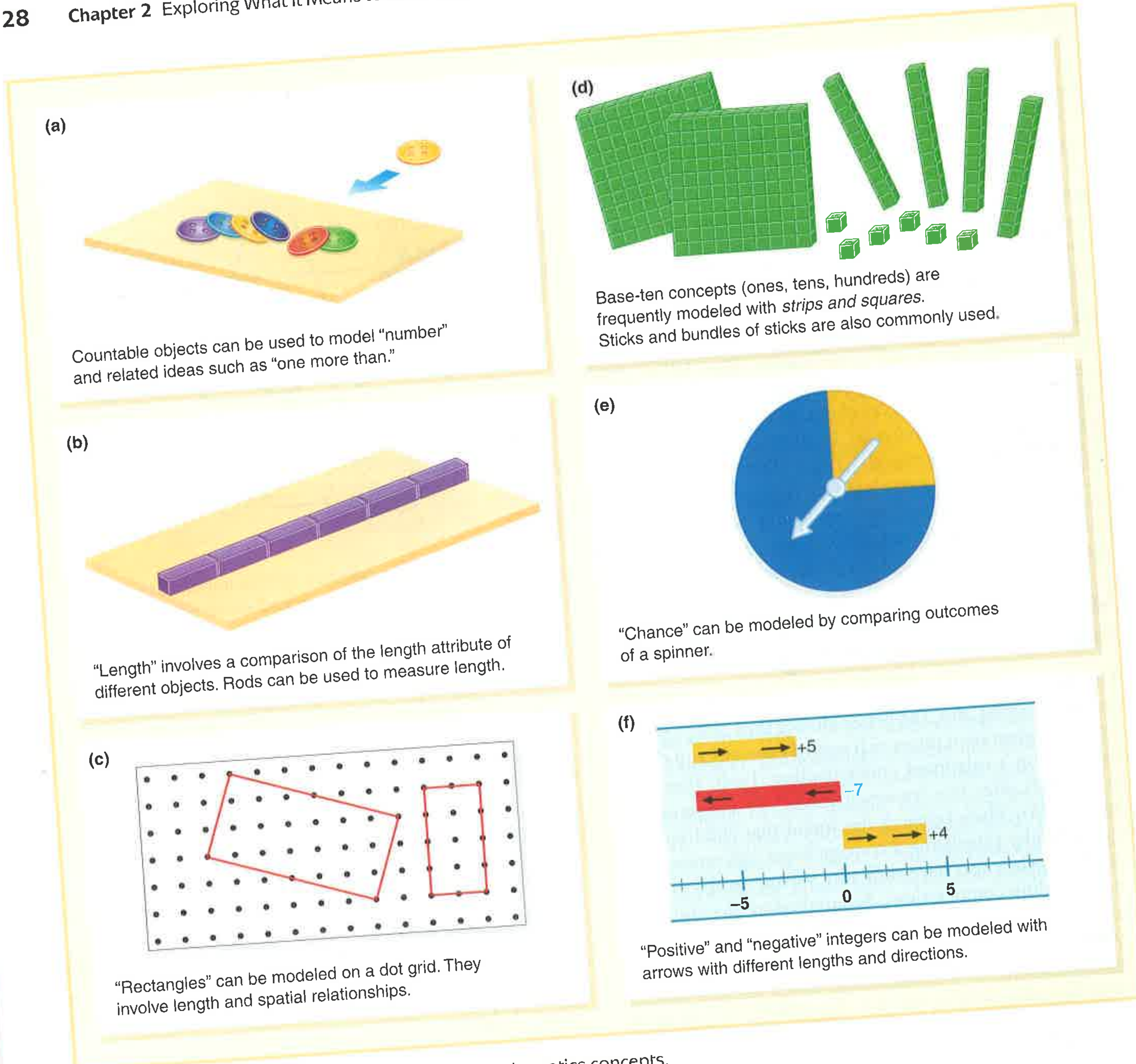
Figure 2.15 Five different representations of mathematical ideas. Translations between and within each can help develop new concepts.

Models can be a testing ground for emerging ideas. It is sometimes difficult for students (of all ages) to think about and test abstract relationships using only words or symbols. For example, to explore the idea of area of a triangle, knowing the area of a parallelogram, requires the use of pictures and/or manipulatives to build the connections. A variety of models should be accessible for students to select and use freely. You will undoubtedly encounter situations in which you use a model that you think clearly illustrates an idea but a student just doesn't get it, whereas a different model is very helpful.

Examples of Models. Physical materials or manipulatives in mathematics abound—from common objects such as lima beans and string to commercially produced materials such as wooden rods (e.g., Cuisenaire rods) and blocks (e.g., Pattern Blocks). Figure 2.16 shows six models, each representing a different concept, giving only a glimpse into the many ways each manipulative can be used to support the development of mathematics concepts and procedures.

STOP

Consider each of the concepts and the corresponding model in Figure 2.16. Try to separate the physical model from the relationship that you must impose on the model in order to "see" the concept.



(a) Countable objects can be used to model "number" and related ideas such as "one more than."

(b) "Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(c) "Rectangles" can be modeled on a dot grid. They involve length and spatial relationships.

(d) Base-ten concepts (ones, tens, hundreds) are frequently modeled with *strips and squares*. Sticks and bundles of sticks are also commonly used.

(e) "Chance" can be modeled by comparing outcomes of a spinner.

(f) "Positive" and "negative" integers can be modeled with arrows with different lengths and directions.

Figure 2.16 Examples of models to illustrate mathematics concepts.

The examples in Figure 2.16 are models that can show the following concepts:

a. The concept of "6" is a relationship between sets that can be matched to the words *one*, *two*, *three*, *four*, *five*, or *six*. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship "one more than."

b. The concept of "measure of length" is a comparison of the length attribute of different objects. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.

c. The concept of "rectangle" includes both spatial and length relationships. The opposite sides are of equal length and parallel and the adjacent sides meet at right angles.

d. The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").

e. "Chance" is a relationship between the frequency of an event's happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.

f. The concept of a "negative integer" is based on the relationships of "magnitude" and "is the opposite of." Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

Ineffective Use of Models and Manipulatives. In addition to not making the distinction between the model and the concept, there are other ways that models or manipulatives can be used ineffectively. One of the most widespread misuses occurs when the teacher tells students, "Do as I do." There is a natural temptation to get out the materials and show children exactly how to use them. Children mimic the teacher's directions, and it may even look as if they understand, but they could be just mindlessly following what they see. It is just as possible to get students to move blocks around mindlessly as it is to teach them to "invert and multiply" mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001).

A natural result of overly directing the use of models is that children begin to use them as answer-getting devices rather than as tools used to explore a concept. For example, if you have carefully shown and explained to children how to get an answer to a multiplication problem with a set of base-ten blocks, then students may set up the blocks to get the answer but not focus on the patterns or processes that can be seen in modeling the problem with the blocks. A mindless procedure with a good manipulative is still just a mindless procedure.

Conversely, leaving students with insufficient focus or guidance results in nonproductive and unsystematic investigation (Stein & Bovalino, 2001). Students may be engaged in conversations about the model they are using, but if they do not know what the mathematical goal is, the manipulative is not serving as a tool for developing the concept.

Technology-Based Models. Technology provides another source of models and manipulatives. There are websites, such as the Utah State University National Library of Virtual Manipulatives, that have a range of manipulatives available (e.g., geoboards, base-ten blocks, spinners, number lines). Virtual manipulatives are a good addition to physical models, as some students will prefer the electronic version; moreover, they may have access to these tools outside of the classroom.

It is important to include calculators as a tool. The calculator models a wide variety of numeric relationships by quickly and easily demonstrating the effects of these ideas. For example, you can skip-count by hundredths from 0.01 (press 0.01 [+] .01 [=], [=], [=] . . .) or from another beginning number such as 3 (press [+] 0.01 [=], [=], [=] . . .). How many presses of [=] are required to get from 3 to 4? Many more similar ideas are presented in Chapter 7.

Connecting the Dots

It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what *doing* mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections among mathematics concepts—connecting the blue dots.

myeducationlab

Go to the Activities and Application section of Chapter 2 of MyEducationLab. Click on Videos and watch the video entitled **"John Van de Walle on Connecting the Dots"** to see him talk with teachers about understanding students' thinking.

Second, you read about learning theory—the importance of having opportunities to connect the dots. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

Finally, you read about understanding—that to have the relational knowledge (knowledge where blue dots are well connected) requires conceptual and procedural understanding, as well as other proficiencies. The problems that you solved in the first section included a focus on concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning, and productive disposition.

This chapter focused on connecting the dots between theory and practice—building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.