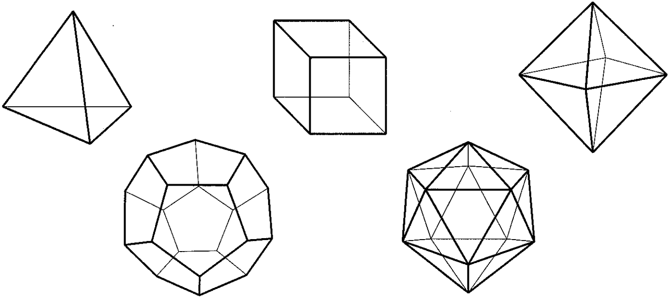
Euler's Formula

For Convex Polyhedra

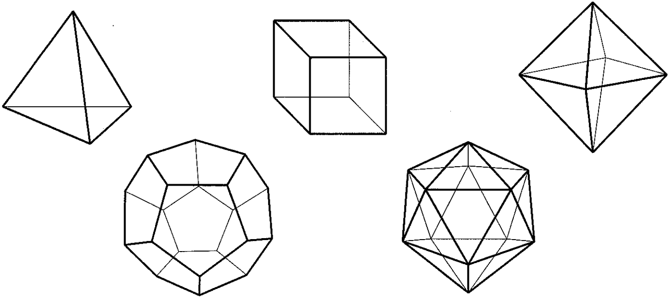


The Platonic Solids

Polyhedron	Faces	Edges	Vertices	Shape of face	Faces per vertex
Tetrahedron					
Cube					
Octahedron					
Dodecahedron					
Icosahedron					

1. What patterns do you see in the table?

For Convex Polyhedra

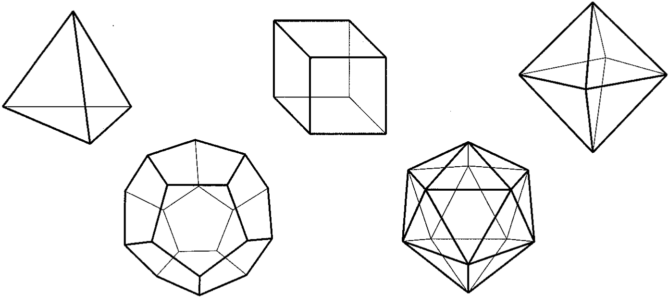


Theorem (Euler's Formula)

The number of vertices V, faces F, and edges E in a convex 3-dimensional polyhedron, satisfy $V + F - E = 2$.

Example: For a cube $V = 8$, $F = 6$, $E = 12$, so $8 + 6 - 12 = 2$

For Convex Polyhedra



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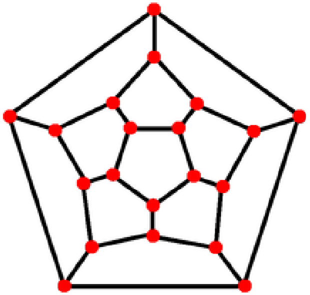
This simple and beautiful result has led to deep work in topology, algebraic topology and theory of surfaces in the 19th and 20th centuries.

It also has great significance in graph theory, computational geometry, and other parts of mathematics.

For Graph Theory

Theorem (Euler's Formula)

If a finite, connected, planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region), then $v + f - e = 2$.



For example, $v = 20$, $f = 12$, $e = 30$, so $20 + 12 - 30 = 2$

Leonard Euler (1707-1783)

- Euler was the first person to notice 'his formula' for 3-D polyhedra.
- He mentioned it in a letter to Christian Goldback in 1750.
- He then published two papers about it and 'attempted' a proof of the formula by decomposing a polyhedron into smaller pieces.
- His proof was incorrect.

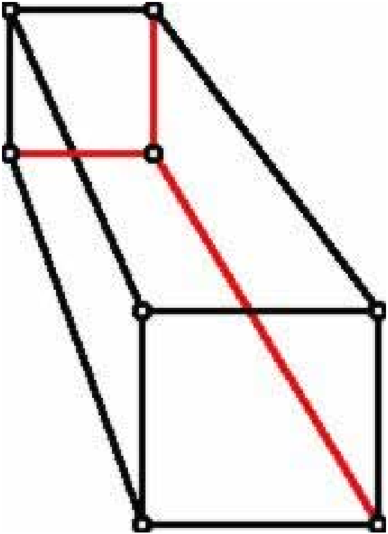
Leonard Euler (1707-1783)

- The first correct proof was given by a French mathematician Adrian Marie Legendre (1752-1833) using metrical methods.
- Although Euler is the ‘father’ of graph theory, he did not make the connection to graph theory.

Augustin Louis Cauchy
(1789-1857)

- Cauchy seems to be the first person who noticed that a convex polyhedron has a planar graph representation.
- He gave an proof in 1811 using a graph theory techniques.
- Now there are at least 17 different proofs of Euler's Formula.

Constructing a graph from a polyhedron



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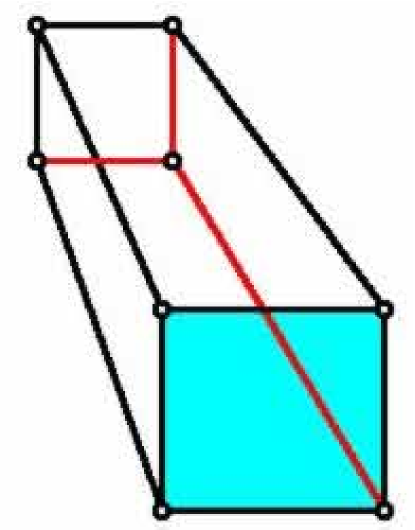

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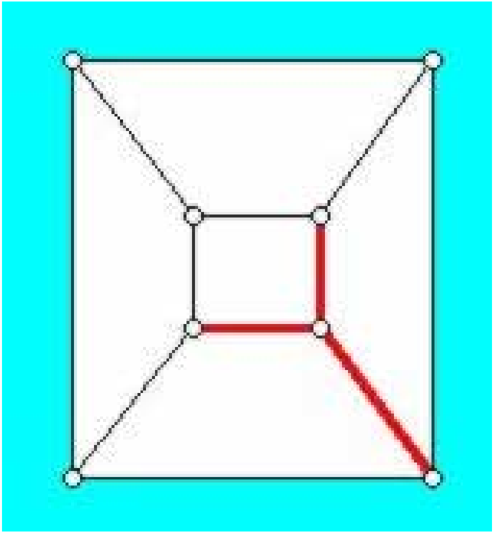
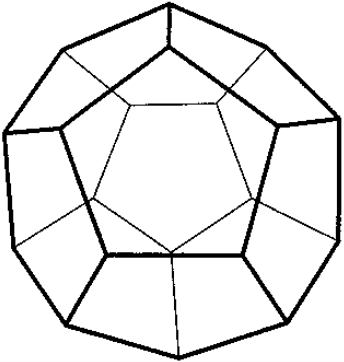
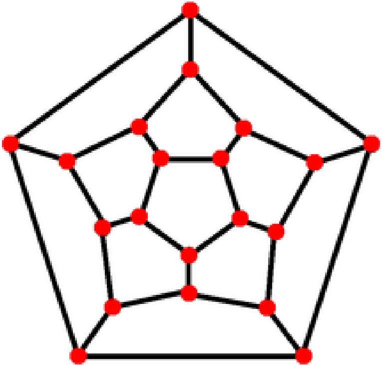
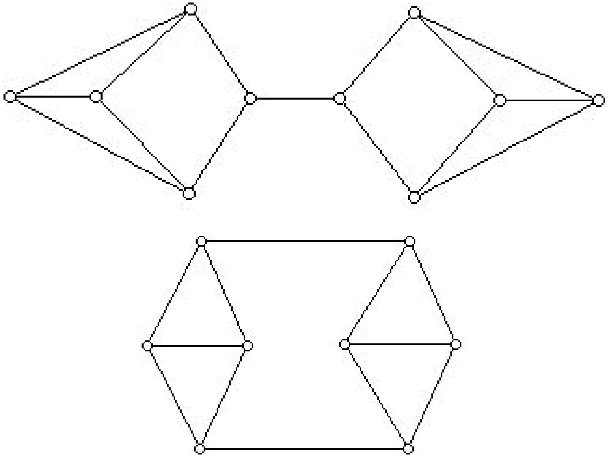






When does a planar graph correspond to a convex 3-dimensional polyhedron?



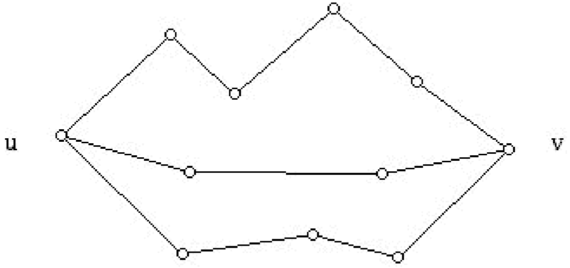
Ernst Steinitz (1871-1928)

- He was a great German geometer and algebraist.
- He completely characterized when a graph ‘has the same structure’ as a convex polyhedron in the early 1900s.
- This theorem allows us to study the theory of 3-dimensional convex polyhedra by drawing diagrams in 2 dimensions.

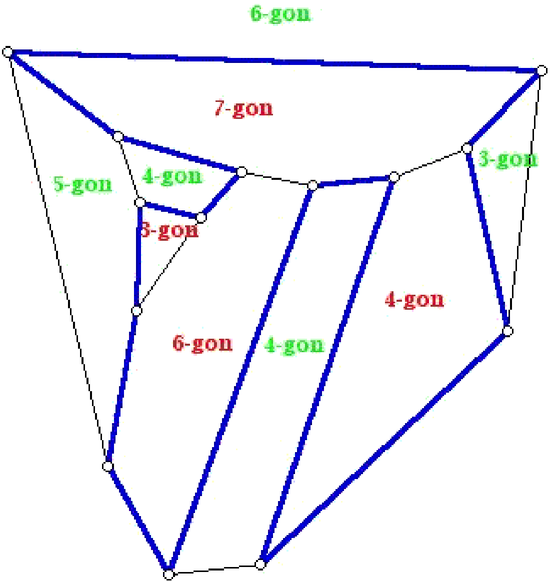
Theorem (Steinitz's Theorem)

A graph G is isomorphic to the vertex-edge graph of a 3-dimensional polyhedron if and only if G is planar and 3-connected.

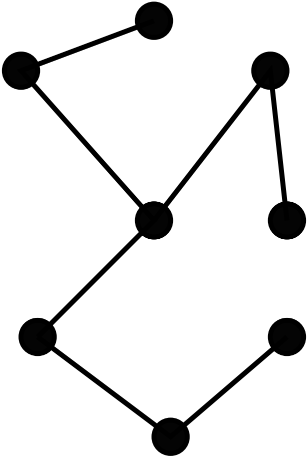
A graph G is 3-connected if for any pair of vertices u and v there are at least three distinct paths between u and v.



Steinitz's Theorem



Proof: $V + F - E = 2$



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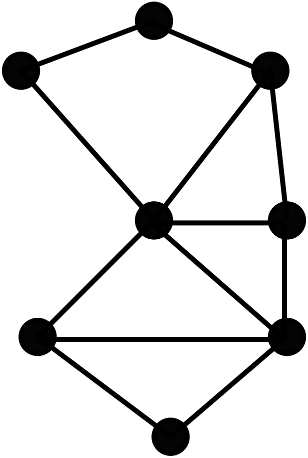

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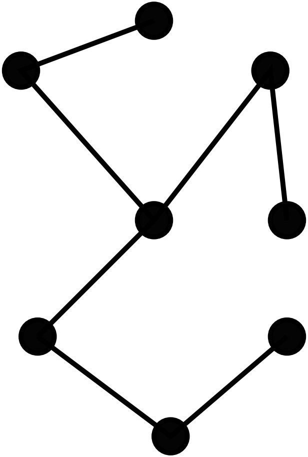

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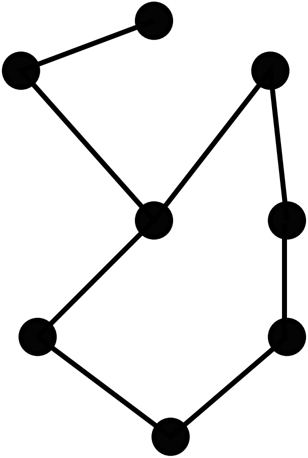

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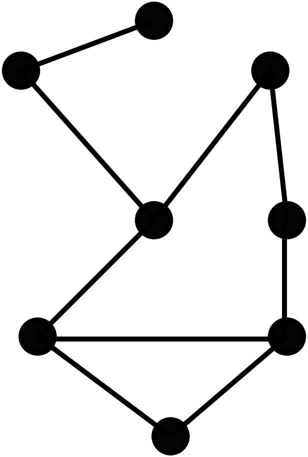

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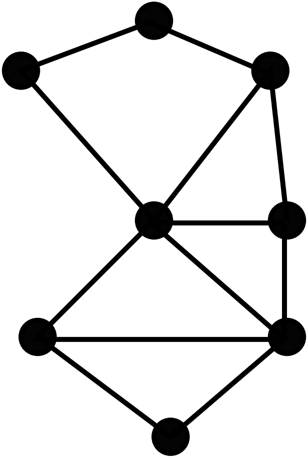

References

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- Malkevitch, J. (2005, January) *Euler's Polyhedral Formula II* AMS Feature Column at www.ams.org/samplings/feature-column